CAE Working Paper #06-13

Portfolio Choice and Asset Prices in an Economy Populated by Case-Based Decision Makers

by

Ani Guerdjikova

August 2006

.

Portfolio Choice and Asset Prices in an Economy Populated by Case-Based Decision Makers

Ani Guerdjikova[1]

This Version: August 2006

I consider an economy populated by case-based decision makers. Consumption can be transferred between the periods by means of a riskless storage technology or a risky asset with i.i.d. dividend payments. I analyze the dynamics of asset holdings and asset prices and identify the influence of the aspiration level, the length of memory and the form of the similarity function. The height of the aspiration level determines whether the economy exhibits constant prices and asset holdings or evolves in a cycle. The length of memory is associated with the ability of the investors to learn the correct distribution of returns, whereas the form of the similarity function influences the willingness of investors to diversify.

Keywords: Case-Based Decision Theory, financial markets, asset pricing.

JEL classification: D81, G12

---

[1] Cornell University, Department of Economics, Uris Hall 462, Ithaca, NY 14853, USA, Tel.: 607/255-4867, Fax: 607/255-2818, e-mail: ag334@cornell.edu. I am indebted to my advisor Juergen Eichberger for his helpful guidance. I would like to thank Jacob Sagi, Hans Gersbach, Verena Liessem, Johannes Becker, Itzhak Gilboa, David Easley, Larry Blume, Ho-Mou Wu and Klaus Ritzeberger for helpful discussions and comments. Financial support by the DFG is greatefully acknowledged.

1 Introduction

The case-based decision theory was proposed by Gilboa and Schmeidler (1995) as an alternative theory for decision making under uncertainty. Differently from the expected utility theory, it models a situation of structural ignorance. Hence, neither states of the world, nor their probabilities are assumed to be known. The sole source of learning is experience, captured by the concept of *memory*. An act is evaluated based on its past performance as well as on the performance of *similar* acts in *similar* circumstances. An *aspiration level* is used as a bench-mark in the evaluation process. It distinguishes results considered satisfactory, i.e. those exceeding the aspiration level, which make an act more attractive, from the unsatisfactory ones.

The case-based decision theory has been applied in several economic contexts[2]. However, up to my knowledge, it has not been used to model decision-making in financial markets. A model of financial markets, in which expected utility maximization is replaced by case-based reasoning is of interest for several reasons. First, it allows to gain a better understanding of the case-based decision theory itself. Second, it contributes to the financial market literature by describing the dynamics of portfolio holdings and asset prices in a market with case-based investors. The analysis of the behavior implied by case-based reasoning allows for comparisons to the predictions of the standard theory, as well as to the empirical findings.

The ”stateless” approach to model behavior in financial markets requires some justification. Since the works of Arrow (1970, p. 98), it has been assumed that the expected utility framework naturally fits the description of an asset in terms of a probability distribution over state-contingent outcomes. However, a thorough consideration of this framework shows that the problem of formulating states of nature in the context of financial market might have no natural solution. Besides the problem of deciding, which payoffs of a security should be considered possible, the question of correlation among the payoffs of different assets arises. Hence, it is not a solution of the problem to identify the states of the world with the payoffs an asset renders[3].

Moreover, in a market environment, payoffs are determined by capital gains, hence by equi-

[2] As for instance, in the consumer theory, Gilboa and Schmeidler (1997, 2001 a), Gilboa and Pazgal (2001), theory of voting, Aragones (1997 a), production theory, Jahnke, Chwolka and Simons (2001), social learning, Blonski (1999), cooperation in games, Pazgal (1997), herding behavior, Krause (2003), choices among lotteries, Gayer (2003).

[3] See Bossert, Pattanaik and Xu (2000, p.296) for a discussion of the problems connected with the construction of states.

librium prices, which themselves depend on the expectations of the market participants. The well known beauty contest used by Keynes (1936) to describe the expectation formation in asset markets illustrates this point. As Arthur (1995, p. 23) notes "[w]here forming expectations means predicting an aggregate outcome that is formed in part from others' expectations, expectation formation can become self-referential. The problem of logically forming expectations then becomes ill-defined, and rational deduction finds itself with no bottom ground to stand upon". Since the case-based decision theory does not rely on the definition of states and state-contingent outcomes or on formation of probabilistic beliefs, it allows to address these problems in a formal model.

Apart from this methodological issue connected with the application of the expected utility theory to model financial markets, the empirical literature has identified multiple violations of the joint hypothesis of expected utility maximization and rational expectations. The approach usually chosen in the literature to address these issues consists in studying the behavior of investors who satisfy the hypothesis of expected utility maximization but have biased beliefs about the distribution of future returns. For instance, Daniel, Hirshleifer and Subrahmanyam (1998) and Gervais and Odean (2001) provide an explanation of excessive trading frequency based on the self-attribution bias. De Long, Shleifer, Summers and Waldmann (1990b), Barberis, Shleifer and Vishny (1998) and Cutler, Poterba and Summers (1990) assume that traders condition their behavior on past returns, which induces positive correlation of asset returns. De Long, Shleifer, Summers and Waldmann (1990a), as well as Shleifer and Vishny (1997) show how noise traders generate arbitrage possibilities in a theoretical model.

In contrast to this work, in this paper the framework of the expected utility theory is completely abandoned and replaced by the framework of the case-based decision theory. The influence of its three major parameters — aspiration level, memory and similarity perceptions — on portfolio holdings and price dynamics is examined.

It is found that investors with relatively low aspiration levels behave in a satisficing manner, choosing constant, but possibly suboptimal portfolios over time. An economy populated by such investors, therefore, exhibits constant prices. In contrast, investors with high aspiration levels constantly switch among the available portfolios causing stochastic or deterministic cycles.

The memory of the individuals plays a crucial role for the dynamic. In order to study the influence of the length of memory on decisions, I analyze two extreme cases — the case of one-period memory and the case of infinite memory. Investors with short memory are unable to learn enough about the possible price and dividend realizations in order to make optimal choices. The behavior of such investors is either stationary but suboptimal, or exhibits cycles. The ability to remember long sequences of realizations per se does not insure optimal behavior in the limit, either. The results of the paper show that long memory leads to optimal behavior, only if the aspiration level is appropriately chosen and the similarity function is convex.

The perceived similarity between portfolios is found to influence choices in a non-trivial way. The curvature of the similarity function determines whether an individual exhibits preferences for diversification. A similarity function that is concave in the Euclidean distance leads investors with relatively high aspiration levels to choose only undiversified portfolios. In contrast, a convex similarity function combined with a high aspiration level implies that an investor will experiment with almost all diversified portfolios in the limit.

The paper is structured as follows. In section 2, I introduce the model of the economy and the case-based decision rule and define the notion of equilibrium. Section 3 analyzes the dynamics of the economy and determines the long-run distribution of the asset-prices for the case of one-period memory. In section 4, the case of long memory is considered. Section 5 discusses the findings in the light of the empirical literature from financial markets and section 6 concludes. The proofs of the results are stated in the appendix.

2 The Economy

Consider an economy consisting of a continuum of investors, uniformly distributed on the interval $[0;1]$. There are two types of investors, $i \in \{1;2\}$. The share of type 1 is denoted by $\theta_1 \in (0;1)$ and assumed constant over time.

There is a single consumption good in the economy. Each investor lives for two periods and consumes only in the second period of his life. Preferences over consumption are represented by a utility function $u(\cdot)$, which is identical for all investors. I assume:

$(A1)$ $u(\cdot)$ is strictly monotonically increasing and continuous in consumption in period two.

The young investors are endowed with one unit of the consumption good. The endowment of the old investors is 0.

There are two assets: a risky asset a and a riskless asset b. The riskless asset b delivers $(1+r)$ units of consumption per unit invested. It is available in perfectly elastic supply at a price of 1.

The supply of a is fixed at 1. Its payoff per unit is $\delta_t \sim^Q \left[\underline{\delta}; \bar{\delta}\right]$ and δ_t is i.i.d. across time. Q is assumed to be continuous and $g(\cdot)$ denotes its density[4]. The price of a at time t is p_t. Short sales are prohibited.

In the spirit of the case-based decision theory, I assume that the investors in the economy have almost no information about the problem they are facing. They do not know the structure of the economy, the process of price formation, or the possible prices and returns of the assets and their distribution. Their information consists of the problem formulation, the set of possible acts and their memory.

2.1 Problems, acts and utility realizations

Each investor solves the problem: "for a given equilibrium price p_t, invest the initial endowment in a portfolio to enable consumption". In general, the description of an investment problem might also depend on multiple factors, such as initial endowment, price level, etc. In this model, however, p_t fully characterizes the market situation. Hence, a decision problem is identified with p_t.

Let α_t^i denote the share of initial endowment invested in a ($\alpha_t^i \in [0;1]$) at time t by an investor of type $i \in \{1;2\}$.

The indirect utility of investor i for a given α_{t-1}^i is:

$$v_t\left(\alpha_{t-1}^i\right) = u\left(\frac{p_t + \delta_t}{p_{t-1}}\alpha_{t-1}^i + \left(1-\alpha_{t-1}^i\right)(1+r)\right).$$

2.2 Aspiration Levels

The aspiration level is the lowest level of utility which renders the investor satisfied with his choice. I assume that the two types of investors differ with respect to their aspiration levels $\bar{u}^1$

[4] Q is interpreted as the objective probability distribution known to an external observer, but not to the investors in the economy. Hence, Q will be irrelevant for the investors' decisions. However, the specification of Q makes it possible to analyze the long-run behavior of the economy.

and $\bar{u}^2$. Typically, the aspiration level would be updated according to the past experience of a decision-maker. I will assume, however that the aspiration levels remain constant over time, and will derive the price and portfolio dynamics depending on the aspiration level. Introducing an updating rule would change the limit results of the process[5]. However, if the updating is relatively slow, the results derived here would still describe the interim dynamics of the process for different ranges of values that $\bar{u}^1$ and $\bar{u}^2$ can assume.

2.3 Similarity

The similarity function is written as

$$s\left((p;\alpha);(p';\alpha')\right) \to [0;1].$$

$s\left((p;\alpha);(p';\alpha')\right)$ can be interpreted as the likelihood assigned to the event that portfolio α bought at price p renders the same utility as portfolio α' bought at price p'.[6]

I use the Euclidean distance between $(p;\alpha)$ and $(p';\alpha')$ as a measure of similarity. Note that the prohibition of short sales and the fixed endowments imply $p_t \in [0;1]$. Hence, $(p;\alpha) \in [0;1]^2$. Hence, s can be defined on $[0;1]^2$. Let

$$s\left((p;\alpha);(p';\alpha')\right) = f\left(\|(p;\alpha)-(p';\alpha')\|\right),$$

with $f(\cdot) \geq 0$ and $f'(\cdot) < 0$. $s(\cdot)$ does not depend on the type of investors. W.l.o.g., I set

$$s\left((p;\alpha);(p;\alpha)\right) = 1^7.$$

2.4 Memory

The memory of an investor describes his information about past utility realizations of available portfolios. Let the memory of an investor of type $i \in \{1;2\}$ consist only of cases $\left(p_t;\alpha_t^i;v_{t+1}\left(\alpha_t^i\right)\right)$

[5] The most common empirically observed aspiration level updating rule is

$$\bar{u}_{t+1} = (1-\beta)\,\bar{u}_t + \beta u_t,$$

where β is a constant which captures the speed of adjustment. With a single act α and an i.i.d. random process u_t, this process is known to converge in expectation to the mean value of $u(\alpha)$. With several acts (preserving the i.i.d. properties of the utility realizations), however, Börgers and Sarin (2000) demonstrate that optimal choice in the limit fails to obtain unless the optimal alternative is dominant or riskless.

[6] The concept closest in spirit to similarity is covariance, which, however, relies on the definition of state-contingent outcomes and their distribution. Hence, its usage does not seem appropriate in a model case-based decisions. Matsui (2000) establishes a connection between the two notions.

experienced by investors of previous generations of his own type[11] and contain the m last cases:

$$M_t^i = \left(\left(p_{t-1}; \alpha_{t-1}^i; v_t\left(\alpha_{t-1}^i\right)\right); \ldots \left(p_{t-m}; \alpha_{t-m}^i; v_{t-m+1}\left(\alpha_{t-m}^i\right)\right)\right)$$

$m \in \{1; 2 \ldots t\}$ parameterizes the length of memory: $m = t$, corresponds to remembering all past cases, $m = 1$ stays for one-period memory. Although these extreme cases do not seem realistic, they allow a tractable analysis and provide an intuition about the influence of the length of memory on the price dynamics.

I assume that only actually observed cases are considered by each investor. This can be justified by the fact that people usually consider their actual experience to be more important than hypothetical cases. Alternatively, the construction of hypothetical cases might be connected with some real or mental costs. For instance, mutual and investment funds do not provide information about all past returns in their brochures, but only a selective past history. The model presented here is quite extreme in that it assumes that hypothetical cases are assigned a weight of 0. If the similarity between hypothetical and actual cases is sufficiently low, the results of this paper would still hold.

In Guerdjikova (2004 b, pp. 204-209), hypothetical cases are introduced and their effect on asset prices and portfolio holdings discussed. I comment on these results and their connection to the current model in section 6 .

2.5 Case-Based Decision Making

Since in period $t = 0$, the memory of the investors is empty, let $\alpha_0 \in (0; 1)$ be the act chosen[12] in period 0 by both types and let $p_0 = \alpha_0$ be the equilibrium price at $t = 0$.

In all consequent periods, the investors choose the portfolio with maximal cumulative utility at the market price. The cumulative utility of a portfolio α for investor i is defined as:

$$U_t^i(\alpha) = \sum_{\tau=t-m}^{t-1} s\left(\left(p_\tau; \alpha_\tau^i\right); \left(p_t; \alpha\right)\right)\left[v_{\tau+1}\left(\alpha_\tau^i\right) - \bar{u}^i\right].$$

Hence, the evaluation of a portfolio α at time t increases, if portfolios considered similar to α

[11] This means that investors do not observe all past choices and realizations, but only those of a given cohort of their predecessors. One possibility to relax this assumption is by introducing social learning, as in Blonski (1999) and Krause (2003).

[12] The results for $\alpha_0 = 1$ and $\alpha_0 = 0$ are qualitatively the same, the interesting case is, however, the one of a diversified initial portfolio. The assumption that α_0 is identical for both types is made for convinience, but would also leave the results qualitatively unchanged.

performed above the aspiration level when bought at prices similar to p_t.

2.6 Equilibrium Paths

Definition 1 Given the initial allocation α_0 and the initial price $p_0 = \alpha_0$, an equilibrium path of the economy is defined as a vector of asset prices $(p_t^*)_{t=0,1,\dots}$ and a vector of portfolios $(\alpha_t^{1*};\alpha_t^{2*})_{t=0,1,\dots}$ chosen by the young investors at t (with $\alpha_0^{1*} = \alpha_0$, $\alpha_0^{2*} = \alpha_0$, $p_0^* = p_0$) such that:

(i) young investors make case-based decisions in each period:

$$\begin{aligned}\alpha_t^{i*} &\in \arg\max_{\alpha\in[0;1]} U_t^i(\alpha) = \\ &= \arg\max_{\alpha\in[0;1]} \sum_{\tau=t-m}^{t-1} s\left((p_\tau;\alpha_\tau^i);(p_t;\alpha)\right)\cdot \\ &\quad \cdot\left[u\left(\left(\frac{p_{\tau+1}+\delta_{\tau+1}}{p_\tau}\right)\alpha_\tau^i + (1+r)\left(1-\alpha_\tau^i\right)\right) - \bar{u}^i\right]\end{aligned}$$

and

(ii) the market for the risky asset is cleared in each period: either $p_t^* > 0$ and satisfies

$$\frac{\theta_1\alpha_t^{1*}(p_t^*) + (1-\theta_1)\,\alpha_t^{2*}(p_t^*)}{p_t^*} = 1$$

or

$$p_t^* = 0 \text{ and } \alpha_t^{1*}(0) + \alpha_t^{2*}(0) = 0.$$

The market clearing condition allows for degenerate equilibria, in which the demand for a and its price are 0. Guerdjikova (2003) shows that such equilibria occur for high values of $\bar{u}^1$ and $\bar{u}^2$. In order to exclude equilibria with $p_t = 0$, I assume that $\bar{u}^1$ is sufficiently low. Hence, the investors of type 1 are always willing to hold a positive quantity of the risky asset.

$(A2)$ Let $\bar{u}^1 < u\left(\frac{\theta_1\alpha_0+\underline{\delta}}{1-\theta_1(1-\alpha_0)}\alpha_0 + (1-\alpha_0)(1+r)\right)$.

$(A2)$ insures that $\alpha_t^{1*} = \alpha_0$ and $p_t \geq \theta_1\alpha_0$ for all $t \geq 1$.

$(A2)$ is a very strong assumption, since it postulates that investors of type 1 never deviate from their initially chosen portfolio. I comment on the effects from dropping $(A2)$ after I state the main results of the paper.

To avoid the discussion of multiple cases, let

$$1 > \bar{\delta} > r > \alpha_0 r > \underline{\delta} \geq 0.$$

3 Price Dynamics with One-Period Memory

Let $m = 1$, hence, the investors only remember the last case observed. Since $\alpha_t^{1*} = \alpha_0$, only the behavior of type 2 needs to be considered. Depending on $\bar{u}^2$, the following results obtain:

Proposition 1 *Assume* $(A1)$ *and* $(A2)$.

1. Let
$$\bar{u}^2 < u\left(\left(1+\frac{\delta}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right). \tag{1}$$
Then, there is an equilibrium path on which $\alpha_t^{1*} = \alpha_0$, $\alpha_t^{2*} = \alpha_0$ and $p_t^* = p_0$ for each $t \geq 1$.

2. Let
$$\bar{u}^2 \in \left(u\left(\left(1+\frac{\delta}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right); u(1+r)\right). \tag{2}$$
Then, on almost all paths of dividend realizations $\tilde{\omega} = (\delta_1; \delta_2 ... \delta_t ...)$, there is an equilibrium path, such that $\alpha_t^{1*} = \alpha_0$, $\alpha_t^{2*} = 0$ and $p_t^* = \theta_1 \alpha_0$ for all $t \geq \bar{t}(\tilde{\omega})$, for some $\bar{t}(\tilde{\omega})$.

3. Let $\bar{u}^2 \in \left(u(1+r); u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-\alpha_0)}\right)\right)$. Then on almost all paths of dividend realizations $\tilde{\omega}$, there is a time $\bar{t}(\tilde{\omega})$, such that for all $t \geq \bar{t}(\tilde{\omega})$ the economy evolves according to a stochastic cycle with two states:
$$\begin{array}{lll} h\text{, with} & \alpha_h^1 = \alpha_0,\ \alpha_h^2 = 1 \text{ and} & p_h = 1-\theta_1(1-\alpha_0) \\ & \text{and} & \\ l\text{, with} & \alpha_h^1 = \alpha_0,\ \alpha_l^2 = 0 \text{ and} & p_l = \theta_1\alpha_0. \end{array}$$
Define $\hat{\delta}$ as
$$\bar{u}^2 = u\left(1+\frac{\hat{\delta}}{1-\theta_1(1-\alpha_0)}\right)$$
and q as:
$$q = \int_{\hat{\delta}}^{\bar{\delta}} g(\delta)\, d\delta.$$
The frequencies with which the two states h and l occur almost surely satisfy:
$$\begin{aligned} \bar{\pi}_h &= \frac{1}{2-q} \\ \bar{\pi}_l &= \frac{1-q}{2-q}. \end{aligned}$$

4. Let $\bar{u}^2 > u\left(1+\frac{\bar{\delta}}{1-\theta_1(1-\alpha_0)}\right)$. Then, on almost all paths of dividend realizations $\tilde{\omega}$, there is a time $\bar{t}(\tilde{\omega})$ such that for all $t \geq \bar{t}(\tilde{\omega})$ the economy evolves in a deterministic cycle of period 2 with two states h and l, as described in 3.

The proposition shows that relatively low aspiration levels (as in parts 1. and 2.) lead to stable prices and asset allocations in the economy. Type 2 is satisfied with the returns of their portfolios and has no incentive to experiment and change portfolio holdings with time.

The proof of part 2. of proposition 1 further demonstrates that for $\alpha_0 < \frac{1}{2}$, the price of a may continually rise for a finite number of periods before it stabilizes at p_l. The price increases in periods of low dividends. Intuitively, when type 2 investors are unsatisfied with the return of a portfolio, they try to switch to a portfolio which is furthest away from the initially chosen one. If $\alpha_0 < \frac{1}{2}$, the switch from $\alpha_{t-1}^2 = \alpha_0$ to $\alpha_t^2 = 1$, causes p_t to rise above p_0 and, hence, the utility derived from α_0 may exceed $\bar{u}^2$. If this is the case, the equilibrium conditions require type 2 to be indifferent among all portfolios and to choose a portfolio $\alpha \in (\alpha_0; 1)$, which implies $p_t > p_0$. This process continues until $\alpha_T^{2*} > \frac{1}{2}$ obtains in some period T. Once α^2 exceeds $\frac{1}{2}$, a sufficiently low realization of δ will force type 2 to switch to b. Afterwards, the economy remains in state l forever.

Conversely, if $\bar{u}^2$ is relatively high (as in parts 3. and 4.), so that type 2 investors are not satisfied with the returns of b and with some of the returns of a, they will permanently switch between the two corner portfolios and the economy will evolve in a cycle.

The results of proposition 1 hinge on $(A2)$. It seems, however, that relaxing $(A2)$ will not significantly change the patterns of the dynamic. Specifically, the conclusion of constant prices and portfolio holdings will still be true as long as both aspiration levels are low relative to the equilibrium payoffs. In contrast, when the aspiration levels of both types of investors are relatively high, so that at least some of the equilibrium payoffs are deemed unsatisfactory, cycles will emerge. These cycles might have up to 4 states, i.e.

$$\left(\alpha_t^1 = \alpha_t^2 = 1\right), \left(\alpha_t^1 = \alpha_t^2 = 0\right), \left(\alpha_t^1 = 1, \alpha_t^2 = 0\right), \left(\alpha_t^1 = 0, \alpha_t^2 = 1\right)$$

and whenever the behavior of a given type is cyclical, this type will only hold corner portfolios. The dynamics of the economy will still follow a Markov process and the frequencies of the states can be computed by finding its invariant distribution. Assumption $(A2)$ is useful in that it allows for an explicit computation of these frequencies and makes it possible to compare the price patterns with empirical findings in section 5.

In proposition 1, the assumption of strictly decreasing similarity function implies that investors with high aspiration levels diversify only for a finite number of periods. As long as the memory is short, this result is independent of the curvature of $s(\cdot;\cdot)$. In contrast, with long memory, the curvature of $s(\cdot;\cdot)$ determines the willingness of the investors diversify.

4 Price Dynamics with Long Memory

In this section, I assume that memory contains all past cases, i.e. $m = t$. The introduction of long memory allows to study the effects of learning on asset prices and portfolio holdings.

I first consider the case of a concave similarity function.

4.1 The Case of Concave Similarity

Let $f' < 0$ and let e denote the Euclidean distance functional. Concavity of the similarity function implies that the matrix

$$d^2 s = f'' de \cdot (de)^T + d^2 e f',$$

is negative definite. Since $e(\cdot;\cdot)$ is concave, this assumptions implies that f should not be too convex.

Denote by $\mu(\alpha \mid p)$ the expected utility from holding portfolio $\alpha \in [0;1]$ at time t, given that the price of α remains constant at $p = p_t = p_{t+1}$:

$$\mu(\alpha \mid p) =: \int_{\underline{\delta}}^{\bar{\delta}} u\left(\left(1 + \frac{\delta}{p}\right)\alpha + (1-\alpha)(1+r)\right) g(\delta)\, d\delta.$$

Let

$$p^*(\alpha) =: \alpha_0 \theta_1 + (1-\theta_1)\alpha \text{ for } \alpha \in [0;1]$$

be the equilibrium price which obtains if $\alpha^{1*} = \alpha_0$ and $\alpha^{2*} = \alpha$. To avoid the discussion of multiple cases, assume:

$$\mu(\alpha_0 \mid p_0) < u(1+r) < \mu(1 \mid p^*(1)) \tag{3}$$

Since the memory of type 2 contains all past cases, their behavior in the long run is determined by the mean of the observed utility realizations computed at equilibrium prices. The concavity of the similarity function implies that there is a $\bar{t}$, such that for all $t \geq \bar{t}$, $\alpha_t^{2*} \in \{0;1\}$. Condition (3), therefore, assumes one possible ordering of the mean utilities of the portfolios actually chosen by type 2.

Proposition 2 *Define $\hat{\delta}$ as:*

$$u\left(\frac{1-\theta_1(1-\alpha_0)+\hat{\delta}}{1-\theta_1(1-\alpha_0)}\right)=\bar{u}^2.$$

Suppose that for some $\zeta>0$, $g(\cdot)$ is continuous w. r. t. the Lebesgue measure and strictly bounded away from 0 *on* $\left[\hat{\delta}-\zeta;\hat{\delta}+\zeta\right]$.

1. If $\bar{u}^2<\mu(\alpha_0\mid p_0)$, the expected time during which type 2 holds α_0 is infinite. Hence, the expected time which the economy spends in the state $(\alpha^1=\alpha_0;\alpha^2=\alpha_0;p=p_0)$ is infinite.

2. If $\bar{u}^2\in(\mu(\alpha_0\mid p_0);u(1+r))$, type 2 chooses either $\alpha_t^{2*}=1$ with frequency 1 or $\alpha_t^{2*}=0$ with frequency 1 a.s. in the limit. Hence, on almost each path, the state of the economy is either $(\alpha^1=\alpha_0;\alpha^2=0;p=p_0)$ with frequency 1 or $(\alpha^1=\alpha_0;\alpha^2=1;p=p^*(1))$ with frequency 1.

3. If $\bar{u}^2\in(u(1+r);\mu(1\mid p^*(1)))$, type 2 chooses $\alpha_t^{2*}=1$ with frequency 1 a.s. in the limit. The state of the economy is a.s. $(\alpha^1=\alpha_0;\alpha^2=1;p=p^*(1))$ with frequency 1.

4. If $\bar{u}^2>\mu(1\mid p^*(1))$, type 2 holds $\alpha_t^{2*}=1$, respectively $\alpha_t^{2*}=0$ with strictly positive frequencies π_h, respectively π_l a.s. in the limit, whereas the frequencies of all other acts are 0. π_h and π_l satisfy:
$$\frac{\pi_h}{\pi_l}=\frac{u(1+r)-\bar{u}^2}{\mu_1^r-\bar{u}^2},$$
where μ_1^r denotes the actual mean utility derived by holding asset a as observed by the investors of type 2. Hence, the economy a.s. evolves according to a stochastic cycle with two states h and l, as described in proposition 1.

The interpretation of the results is similar to those derived with one-period memory. The main difference consists in the fact that with long memory, the investors of type 2 whose aspiration level lies between the maximal achievable mean utilities of the undiversified portfolios will learn to choose the one with the higher mean utility. Hence, long memory combined with a correctly chosen aspiration level enhances optimal behavior. Nevertheless, with a concave similarity function investors do not choose diversified portfolios make suboptimal decisions in the limit. The result is due to the fact that similarity does not decrease sufficiently fast in the Euclidean distance. Hence, an investor who eventually observes only the (unsatisfactory) realizations of the corner portfolios tends to overestimate their negative impact on the diversified portfolios.

The next step of the analysis consists in relaxing the assumption of a concave similarity function.

4.2 The Case of Non-Concave Similarity Function

Note that the similarity function which has a maximum at $((p;\alpha);(p;\alpha))$ cannot be convex everywhere. We can, however assume:

$(A3)$ The matrix

$$f''de\cdot(de)^T+d^2ef'$$

is positive semidefinite for all $(\alpha;p)\neq(\alpha';p')$.

$(A3)$ implies that for a given $(\alpha;p)$, $s((\alpha;p);(\alpha';p'))$ is convex on any set $\hat{A}\subset[0;1]^2$ such that $(\alpha;p)\notin\hat{A}$.

$(A3)$ finds empirical support in the literature. Shepard (1962) (a), (b) and Kruskal (1964) (a) and (b) suggest a method which allows them to construct the similarity function as a monotonic function of the Euclidean distance on a space which is derived from the observed data and estimate its exact shape. Shepard (1987, p. 1318) depicts the estimates of a similarity function for twelve different experiments, most of them conducted with humans, in which the ability to distinguish between different stimuli (e.g. sizes, lightness, etc.) was tested[13]. The results strongly suggest that the similarity function is convex in the Euclidean distance and resembles an exponential function, which justifies assumption $(A3)$. However, it would be of interest to test whether similar patterns obtain in economic experiments.

Let

$$C_t^2(\alpha)=\left\{\tau<t\mid\alpha_\tau^{2*}=\alpha\right\}$$

denote the set of past periods in which type 2 has chosen α and let $|C_t^2(\alpha)|$ be the cardinality of this set.

Proposition 3 *Assume* $(A3)$. *If for all* $\alpha\in[0;1]$,

$$\mu(\alpha\mid p^*(\alpha))<\bar{u}^2,$$

then there is no $x\in(0;1)$ *such that for all portfolios* $\alpha\in B_x(\epsilon)$ *(where* $B_x(\epsilon)$ *is an open ball*

[13] Unfortunately, empirical results on the form of the similarity function are sparse in economics. The few exceptions of which I am aware, are the works of Buschena and Zilberman (1995, 1999) and Zizzo (2002). Their findings show that similarity between acts is negatively correlated to the Euclidean distance between payoffs and influences decisions under risk.

with radius ϵ around x), $|C_t^2(\alpha)| = 0$ obtains. For all α and $\alpha' \in A$

$$\lim_{t\to\infty} \frac{U_t^2(\alpha)}{U_t^2(\alpha')} = 1$$

a.s. holds.

Proposition 4 *Assume $(A3)$. If for some $\alpha \in [0;1]$,*

$$\mu(\alpha \mid \alpha_0\theta_1 + \alpha(1-\theta_1)) > \bar{u}^2,$$

then the investors of type 2 a.s. choose a portfolio

$$\alpha^* \in \tilde{A} = \left\{\alpha \in [0;1] \mid \mu(\alpha \mid \alpha_0\theta_1 + \alpha(1-\theta_1)) > \bar{u}^2\right\}$$

with frequency 1 in the limit.

The aspiration level determines whether or not the economy will converge to a stationary state. Whereas with a relatively low aspiration level, type 2 eventually learns to choose one of the portfolios they consider satisfactory, a high aspiration level implies that type 2 constantly switches among the available portfolios and the economy never reaches a stationary point. Proposition 4 does not insure the choice of an optimal portfolio, but it shows that for an appropriately chosen $\bar{u}^2$, the optimal choice can be approximated with an arbitrary precision.

The results derived in this section allow us to differentiate between the influence of the form of the similarity function and the influence of the aspiration level. The curvature of the similarity function determines whether investors with high aspiration levels will be willing to diversify. Whereas a concave similarity function implies that the investors of type 2 hold only undiversified portfolios after switching away from α_0, with a non-concave similarity function this result is no longer valid. On the contrary, a relatively high aspiration level combined with a convex similarity function implies that type 2 will explore the whole space of available portfolios and will hold a diversified portfolio a.s. with frequency 1 in the limit.

5 Implications for Portfolio Holdings and Asset Prices

The results of the last section have shown that it is possible to identify conditions under which case-based investors make almost optimal choices in the limit. In general, however, the behavior of case-based investors differs from the predictions of the standard models. Hence, case-based investors can induce asset price patterns which are documented in the empirical literature and which are viewed as inconsistent with expected utility hypothesis combined with rational expectations. In this section, I discuss how the findings of the paper relate to some of the paradoxes observed in financial markets such as the presence of arbitrage opportunities, bubbles, limited market participation and underdiversification.

5.1 Arbitrage Possibilities, Bubbles and Excessive Volatility

The notion of arbitrage is fundamental to the theory of asset pricing. Unused arbitrage possibilities are nevertheless found in real and experimental markets, see Rosenthal and Young (1990), Lamont and Thaler (2001), Shleifer (2000, Chapter 3), Oliven and Rietz (1995) and Rietz (1998).

The fact that states of nature are not defined in the case-based decision theory means that objectively given arbitrage possibilities might not be perceived by individual investors and remain unused. Consider, e.g. the case of short memory and low aspiration level. If the share of the investors of type 1, θ_1 is relatively low,

$$\underline{\delta} \geq r\theta_1\alpha_0$$

and (2) holds, the stationary state of the economy exhibits arbitrage possibilities. Obviously, the same effect can arise with long memory, but relatively low aspirations, as in part 2 of proposition 2.

The empirical literature documents bubbles and deviations of asset prices from fundamental values, see Kindleberger (1978) and Sunder (1995). Note that in the model presented here, there is no population growth, which (combined with $r > 0$) implies that the economy is dynamically efficient and, hence, a bubble can not be an equilibrium under rational expectations. Nevertheless, the proof of part 2 of proposition 1 demonstrates that a bubble can emerge and burst in this economy. If the fundamental value of a (defined as usual with respect to the actual distribution of returns and using $u(\cdot)$ as a von Neumann Morgenstern utility index) lies between p_l and 1, the risky asset will be overvalued during the bubble and undervalued after the bubble bursts.

Note further that in the model presented in this paper the price under rational expectations would be constant. Nevertheless, for relatively high values of $\bar{u}^2$, we observe excessive price volatility[14] due to changes in α_t^2. The price fluctuations have a greater amplitude, the higher the value of $1 - \theta_1$, i.e. the relative share of type 2. Moreover, type 2 investors switch from $\alpha = 0$ to $\alpha = 1$ in periods of high prices ($p_t = p_h$) and switch from $\alpha = 1$ to $\alpha = 0$ in periods of low prices ($p_t = p_l$), hence they trade at a loss.

[14] See Shiller (1981, 1990) for empirical evidence on excessive volatility.

5.2 Financial Market Participation

Empirical studies show that a large proportion of the consumers in the US do not hold risky assets, see e.g. Mankiw and Zeldes (1991). This is usually explained by the presence of transaction costs and liquidity needs, see Allen Gale (1994), Williamson (1994). Unfortunately, these two factors provide merely a partial explanation of the empirical evidence. Moreover, although the effect is more prevalent among the poor parts of the population, a significant proportion (47,7%) of the rich households abstain from holding risky assets as well. This last fact can hardly be justified by prohibitively high participation costs or liquidity needs.

Another explanation of the limited participation paradox stems from Cao, Wang and Zhang (2005), who use perceived ambiguity about the distribution of returns. Limited participation results in their model when the variance of the perceived ambiguity in the population is very high. Only investors whose perceived ambiguity is small enter the market. However, their results predict that limited participation would lead to undervaluation of the risky asset, contrary to the equity premium puzzle documented by Mehra and Prescott (1985).

The model analyzed in the previous sections provides a different criterion according to which participating investors can be distinguished from those who do not hold risky assets: their aspiration level. With short memory, investors with low aspiration levels (case 1 of proposition 1) will hold risky assets, independently of the current market prices. Investors with high aspiration levels, in contrast, (case 3 of proposition (1)) would only participate in the asset market if both prices and dividends are high and leave the market, once the dividends become low and returns fall below their aspiration level. If such investors are present, market participation increases during market booms and drops after market crashes. Investors with intermediate aspiration levels, (case 2 of proposition 1) will never be active in the market for risky assets in the long-run.

In the case of long memory, two factors determine market participation: the aspiration level and the curvature of the similarity function. As long as the similarity function is concave, the results for market participation are almost identical to those with short memory. In contrast, when assumption $(A3)$ is satisfied, investors with high aspiration levels always participate in the market for risky assets. Hence, full market participation obtains.

5.3 The Equity Premium Puzzle

Market participation is often connected to the equity premium puzzle described by Mehra and Prescott (1985). Empirical studies show that the coefficient of risk aversion needed to justify the excessive returns on risky assets (as compared to bonds) is much lower when limited market participation is taken into account, see Attanasio, Banks and Tanner (2002) and Vissing-Jorgenssen (2002). Vissing-Jorgenssen and Attanasio (2003) use the Epstein-Zin type of preferences which allow them to obtain even better results by separating intertemporal preferences from risk aversion.

As will be shown in subsection 5.4, the case-based decision theory allows for a similar separation between the curvature of the utility function $u(\cdot)$ and preferences for diversification and, therefore, produces effects similar to those present in Vissing-Jorgenssen and Attanasio (2003). Intuitively, keeping the aspiration level constant, varying the curvature of $u(\cdot)$ changes the mean utility which could be potentially derived from holding diversified portfolios as compared to non-diversified ones. At the same time, whether the decision-maker will actually choose diversified portfolios is determined by the curvature of his similarity function. Hence, even investors with a concave utility $u(\cdot)$ might fail to choose a diversified portfolio.

The observed average return of the risky asset at a given point in time is given by:

$$\frac{\sum_{t=0}^{T}\left(\frac{p_{t+1}^{*}}{p_{t}^{*}}+\int_{\underline{\delta}}^{\bar{\delta}}\frac{\delta}{p_{t}^{*}}dQ\left(\delta\right)\right)}{T}.$$

For the cases, in which $p_t^* = p^*$ for all t, we can write the average excess return of the risky asset over the bond as:

$$1+\int_{\underline{\delta}}^{\bar{\delta}}\frac{\delta}{p^{*}}dQ\left(\delta\right)-\left(1+r\right)$$

When type 2 does not participate in the market for the risky asset in the limit, and θ_1 is sufficiently low,

$$1+\int_{\underline{\delta}}^{\bar{\delta}}\frac{\delta}{p^{*}}dQ\left(\delta\right)-\left(1+r\right)=1+\int_{\underline{\delta}}^{\bar{\delta}}\frac{\delta}{\theta_{1}\alpha_{0}}dQ\left(\delta\right)-\left(1+r\right)>0$$

obtains on the equilibrium path. Note, that we can obtain this result for any utility function $u(\cdot)$ by just varying θ_1, $\bar{u}^2$ and $s(\cdot;\cdot)$. In contrast, in a model with expected utility maximizers, the excess return would be uniquely determined by the curvature of their von Neumann Morgenstern utility index.

Suppose now that people behave as expected utility maximizers when faced with choices over lotteries and as case-based decision makers when facing choices for which states of the worlds and probabilities are not explicitly specified. In both cases, however, they use the same function $u(\cdot)$ to evaluate outcomes. Suppose further that we measure the risk-aversion of a population of decision -makers for choices over lotteries in an experiment and find them to be risk-neutral. Nevertheless, we may find that these decision-makers demand a non-zero risk premium when trading in an asset market. Especially, if $u(\cdot)$ is linear and condition (2) (for the case of short memory) or the condition of part 2 of proposition 2 (for the case of long memory) hold, the risky asset is undervalued relative to the bond. Observationally, this would mimic the equity premium puzzle observed by Mehra and Prescott (1985) for appropriately chosen values of θ_1.

5.4 Preferences for Diversification

Differently from the expected utility theory, which does not allow for a separation between decreasing marginal utility and risk aversion, case-based decision-makers may exhibit decreasing marginal utility (concave $u(\cdot)$) and still prefer to hold an undiversified portfolio in the limit if their similarity function is concave. Underdiversification has been recorded in the empirical literature, see Tesar and Werner (1995), Coval and Moskowitz (1999) and Barber and Odean (2000). The last paper documents that most of the investors engage in high frequency trading, while holding only a small number of different assets in their portfolios (about 4). This would be consistent with the assumption of a concave similarity function combined with a relatively high aspiration level, as in part 4 of proposition 2.

The preferences of a case-based decision-maker are captured by the cumulative utility he assigns to different acts. In general, they vary over time. To derive meaningful statements about the willingness to diversify, I work with the limit preferences as $t \to \infty$. Then, $\alpha_1 \sim \alpha_2$ will correspond to

$$\lim_{t\to\infty} \frac{U_t^i(\alpha_1)}{U_t^i(\alpha_2)} = 1.$$

Definition 2 Preference for diversification obtain if for all α_1 and $\alpha_2 \in [0;1]$ such that

$$\lim_{t\to\infty} \frac{U_t^i(\alpha_1)}{U_t^i(\alpha_2)} = 1$$

and all $\beta \in [0; 1]$,

$$\frac{\lim_{t\to\infty} U_t^i \left(\beta\alpha_1 + (1-\beta)\,\alpha_2\right)}{\lim_{t\to\infty} U_t^i \left(\alpha_1\right)} \geq 1,$$

if both the numerator and the denominator converge to $+\infty$ and

$$\frac{\lim_{t\to\infty} U_t^i \left(\beta\alpha_1 + (1-\beta)\,\alpha_2\right)}{\lim_{t\to\infty} U_t^i \left(\alpha_1\right)} \leq 1,$$

if the numerator and the denominator converge to $-\infty$.

Note that the definition of preference for diversification depends (through U_t^i) on the chosen equilibrium path. However, it will be shown that the emergence of preferences for diversification will depend only on the aspiration level and on the form of the similarity function and not on the specific path ω.

The following corollaries obtain:

Corollary 5 [15]*Assume* $(A3)$. *On any equilibrium path, both types of investors exhibit preference for diversification.*

Corollary 6 *Suppose that the similarity function is concave. On any equilibrium path, type 1 exhibits preferences for diversification. Type 2 exhibits preferences for diversification if and only if* $\bar{u}^2 < \mu\,(1 \mid p^*\,(1))$.

6 Conclusion

The present paper has analyzed the dynamics of an OLG economy with case-based investors. The main findings concern the influence of memory, similarity perceptions and aspiration levels on portfolio holdings and asset prices.

One of the assumptions of the paper concerns the constant aspiration level of the investors. On page 147 of their book, Gilboa and Schmeidler (2001 b) state that: ”...when the environment is more or less fixed, and the situation may be modeled as a repeated choice problem, case-based decision makers do appear to be a little too naive and myopic”. Further on, they remark that this myopic behavior can be attributed to constant aspiration levels and proceed to define an adaptation process which insures optimal behavior in the limit for i.i.d. environments with a single decision-maker. One might, therefore, conjecture that using the same adaptation process

[15] The finding that the curvature of the similarity function determines the preferences for diversification is consistent with the results of Nehring and Puppe (1999). In a different setting, they compute the similarity function corresponding to preferences for diversity over acts situated on a one-dimensional simplex. They conclude that preference for diversity implies a similarity function which is convex in the Euclidean distance.

in an i.i.d. economy would lead to optimal behavior and eliminate the phenomena described above.

In Guerdjikova (2005), I show that the optimality result derived by Gilboa and Schmeidler (2001 b) strongly depends on the form of the similarity function. For instance, a concave similarity function in combination with the suggested updating rule implies the choice of the best corner alternative in the limit, a result very similar to the one derived in this paper. In general, a rule similar to the one suggested by Gilboa and Schmeidler (2001 b), leads to almost optimal decisions[16] only if the similarity function is sufficiently concave near the identity and sufficiently convex near the corners of the simplex. It is an interesting research question how the results derived here would change if such similarity functions were used. However, this might make the model computationally intractable.

Moreover, the results derived by Gilboa and Schmeidler (2001 b) and Guerdjikova (2005) hold only for individual decision problems. Even in a representative consumer economy their validity is jeopardized by the non-i.i.d. structure of the decision-making process and, hence, of the price- and return-processes. Hence, there is no reason to believe that the suggested adaptation process would lead to optimal results in this setting.

Furthermore, the results derived here suggest that the behavior of the economy remains qualitatively identical for whole ranges of values of $\bar{u}^2$. Hence, the paper might be seen as describing the short-run behavior of the economy in which the aspiration level is adapted sufficiently slowly.

The paper also makes the seemingly strong assumption that the memory of the investors in the economy consists only of cases observed in the past by investors with the same aspiration. It might seem that working with memory containing all past returns would lead to an equilibrium where investors choose optimal portfolios and the price of the risky asset coincides with its price under rational expectations. In Guerdjikova (2004 b, pp. 204-209), I show that adding hypothetical cases might fail to enhance learning. Allowing investors to observe all hypothetical past cases leads to an economy with a representative investor (independently of the aspiration levels of the different types)[17]. In the short-run, this gives rise to 0-asset prices or to situations, in which the bond is not demanded. In the long-run, the observation of hypothetical cases might

[16] I.e., satisficing decision-making on a set with arbitrarily small measure.
[17] Note that no similarity considerations are needed once the investor observes the past outcomes of all portfolios.

lead to suboptimal choices in cases in which investors using a smaller number of cases learn to choose optimally.

The current model has shown that empirically observed patterns such as unused arbitrage possibilities, limited market participation, the equity premium puzzle, bubbles, excessive volatility and failure to diversify can emerge in a market populated by case-based investors.

Of course, such an explanation of the empirical observations would only be meaningful if it could be shown that case-based investors are able to survive in the presence of rational investors in the market. This is done in Guerdjikova (2004 a), where it is demonstrated that even case-based investors with one-period memory are able to survive and to influence asset prices in the presence of expected utility maximizers.

Appendix

The following notation is convenient and will be used in the course of the proofs. Denote by

$$V_t^2(\alpha) = \sum_{\tau \in C_t^2(\alpha)} u\left(\left(\frac{p_{\tau+1}+\delta_{\tau+1}}{p_\tau}\right)\alpha + (1+r)(1-\alpha)\right) - \bar{u}^2,$$

where

$$C_t^2(\alpha) = \left\{\tau \leq t-1 \mid \alpha_t^{2*} = \alpha\right\}.$$

Proof of proposition 1:

Part 1

The maximal (single period) decline in p_t^* given $\alpha_t^{1*} = \alpha_0$ is

$$\frac{p_t}{p_{t-1}} = \frac{\theta_1\alpha_0}{1-\theta_1(1-\alpha_0)}$$

$(A2)$ insures that even in this case and with $\delta_t = \underline{\delta}$,

$$\begin{aligned} U_t^1(\alpha_0) &= u\left(\frac{\theta_1\alpha_0+\underline{\delta}}{1-\theta_1(1-\alpha_0)}\alpha_0 + (1-\alpha_0)(1+r)\right) - \bar{u}^1 \geq \\ &\geq \left[u\left(\frac{\theta_1\alpha_0+\underline{\delta}}{1-\theta_1(1-\alpha_0)}\alpha_0 + (1-\alpha_0)(1+r)\right) - \bar{u}^1\right](s(\alpha_0;\theta_1\alpha_0);(\alpha;\theta_1\alpha_0)) \end{aligned}$$

for all $\alpha \in [0;1]$. Hence, $\alpha_t^{1*} = \alpha_0$ for all $t \geq 1$.

By (1), we have:

$$\begin{aligned} U_1^2(\alpha) &= s((p_0;\alpha_0);(p_0;\alpha)) \cdot \\ &\cdot \left[u\left(\left(\frac{p_0+\delta_1}{p_0}\right)\alpha_0 + (1+r)(1-\alpha_0)\right) - \bar{u}^2\right] > 0 \end{aligned}$$

and the strict monotonicity of $s(\cdot;\cdot)$ implies $U_1^2(\alpha_0) > U_1^2(\alpha)$ for all $\alpha \neq \alpha_0$. Hence, $\alpha_1^{2*} = \alpha_1^{1*} = \alpha_0$ and $p_1^* = p_0$ constitute an equilibrium for each $\delta_1 \in \left[\underline{\delta}; \bar{\delta}\right]$.

By induction, the same result holds for each $t \geq 1$, hence $(\alpha_0; \alpha_0; p_0)$ is a stationary state of the economy.■

Part 2

By $(A2)$, $\alpha_t^{1*} = \alpha_0$ for all $t \geq 1$. Define $\tilde{\delta}$ as[18]:

$$u\left(\left(1 + \frac{\tilde{\delta}}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right) = \bar{u}^2$$

Note that for $\delta \geq \tilde{\delta}$

$$u\left(\left(1 + \frac{\delta}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right) = u\left(1 + \delta + (1-\alpha_0)r\right) \geq \bar{u}^2,$$

hence, $\alpha_0 = \arg\max_{[0;1]} U_t^2(\alpha)$ and, therefore, as in the proof of part 1 of proposition 1, $\alpha_t^{2*} = \alpha_0$ and $p_t^* = p_0$ is an equilibrium[19] for all t such that $\delta_\tau \geq \tilde{\delta}$ for all $\tau \leq t$. Let $t' = \min\left\{t \mid \delta_t < \tilde{\delta}\right\}$. t' is a.s. finite, but its value depends on the chosen path of dividend realizations $\tilde{\omega}$[20]. If $\alpha_{t'}^2 = \alpha_0$,

$$U_{t'}^2(\alpha_0) = v_{t'}(\alpha_0) - \bar{u}^2 \leq u\left(1 + \delta_{t'} + (1-\alpha_0)r\right) - \bar{u}^2 < 0$$

Since $s\left((\alpha; p_{t'}); (\alpha'; p_{t'})\right)$ is strictly decreasing in $|\alpha - \alpha'|$, type 2, who takes $p_{t'}$ as given, choose $\alpha_{t'}^2 = 1$ if $\alpha_0 < \frac{1}{2}$ and $\alpha_{t'}^2 = 0$ if $\alpha_0 > \frac{1}{2}$.

Case 1: $\alpha_0 \geq \frac{1}{2}$.

Then, $p_{t'}^* = \theta_1\alpha_0$ and $\alpha_{t'}^{2*} = 0$ constitute an equilibrium. After t',

$$v_t\left(\alpha_{t'}^{2*}\right) = u(1+r) > \bar{u}$$

for all t and p_t^*. Hence, the state $(\alpha^1 = \alpha_0; \alpha^2 = 0; p = \theta_1\alpha_0)$ is stationary.

---

18 Observe that since

$$u\left(\left(1 + \frac{\delta}{p_0}\right)\alpha_0 + (1-\alpha_0)(1+r)\right) = u\left(1 + \delta + (1-\alpha_0)r\right) \leq \bar{u}^2 < u(1+r)$$

for $\delta < \tilde{\delta}$, it follows that

$$1 + \delta + (1-\alpha_0)r < 1 + r$$

and therefore that for each $\delta < \tilde{\delta}$

$$\delta < \alpha_0 r < r.$$

Hence, for $\tilde{\delta} \in \left(\underline{\delta}; \bar{\delta}\right)$ to hold, $\underline{\delta} < \alpha_0 r$ must be satisfied. If this assumption is violated, no such $\tilde{\delta}$ exists.

19 Since $\alpha_t^{1*} = \alpha_0$ holds for all t, I will use the phrase "α_t^{2*} and p_t^* constitute an equilibrium" instead of "$\alpha_t^{1*} = \alpha_0$, α_t^{2*} and p_t^* constitute an equilibrium".

20 Similarly, all period numbers introduced hereafter depend on the realized dividend path $\tilde{\omega}$. I neglect this dependence in the notation for convenience.

Case 2: $\alpha_0 < \frac{1}{2}$.

At $p_{t'} = \alpha_0$, $\arg\max_{[0;1]} U^2_{t'}(\alpha) = 1$. If $\alpha^{2*}_{t'} = 1$, $p^*_{t'} = \theta_1\alpha_0 + (1-\theta_1)$ must hold. Hence, if

$$v_{t'}(\alpha_0) = u\left(\frac{\theta_1\alpha_0 + (1-\theta_1) + \delta_{t'}}{p_0}\alpha_0 + (1-\alpha_0)(1+r)\right) > \bar{u}^2,$$

$\alpha^{2*}_{t'} = 1$ and $p^*_{t'} = \theta_1\alpha_0 + (1-\theta_1)$ constitute an equilibrium.

However, if

$$u\left(\frac{\theta_1\alpha_0 + (1-\theta_1) + \delta_{t'}}{p_0}\alpha_0 + (1-\alpha_0)(1+r)\right) > \bar{u}^2, \tag{4}$$

$$U^2_{t'}(\alpha_0) > 0$$

and, therefore, $\alpha^2 = 1$ is not optimal given $p_{t'} = \theta_1\alpha_0 + (1-\theta_1)$. Hence, the equilibrium $p^*_{t'}$ and $\alpha^{2*}_{t'}$ must satisfy:

$$\begin{aligned} U^2_{t'}(\alpha_0) &= u\left(\frac{p^*_{t'} + \delta_{t'}}{p_0}\alpha_0 + (1-\alpha_0)(1+r)\right) - \bar{u}^2 = \\ &= s\left((p^*_{t'};\alpha);(p_0;\alpha_0)\right) U^2_{t'}(\alpha_0) = U^2_{t'}(\alpha) = 0, \end{aligned}$$

for all $\alpha \in [0;1]$ and

$$p^*_{t'} = \theta_1\alpha_0 + (1-\theta_1)\,\alpha^{2*}_{t'}.$$

By $(A1)$, such $p^*_{t'}$ and $\alpha^{2*}_{t'}$ exist and are unique. Furthermore, $\alpha^{2*}_{t'} \in (\alpha_0;1)$ and

$$p^*_{t'} \in (p_0; \theta_1\alpha_0 + (1-\theta_1)).$$

Again, two cases can occur. If $\alpha^{2*}_{t'} \geq \frac{1}{2}$, then $\alpha^{2*}_{t''} = 0$ obtains a.s. for some $t'' = \min\left\{t > t' \mid \delta_t < \tilde{\delta}\right\}$, as shown above. If $\alpha^{2*}_{t'} < \frac{1}{2}$ construct $\alpha^{2*}_{t''}$ in the same manner as $\alpha^{2*}_{t'}$. Obviously, $\alpha^{2*}_{t''} \in (\alpha^{2*}_{t'};1)$. Repeat the same procedure n times as long as $\alpha^{2*}_{t^n} < \frac{1}{2}$. Note that since

$$u\left(\frac{p^*_{t^k} + \delta_{t^k}}{p^*_{t^{k-1}}}\alpha^{2*}_{t^{k-1}} + (1-\alpha^*_{t^{k-1}})(1+r)\right) = \bar{u}^2$$

and

$$p^*_{t^{k-1}} = \theta_1\alpha_0 + (1-\theta_1)\,\alpha^{2*}_{t^{k-1}},$$

it follows that p_{t^k} is given by

$$p_{t^k} = p_{t^{k-1}}\frac{\varpi - (1+r)}{p_{t^{k-1}} - \theta_1\alpha_0}p_{t^{k-1}}(1-\theta_1) + (1+r)\,p_{t^{k-1}} - \delta_{t^k} \tag{5}$$

for all $k = 1...n$, where $\varpi = u^{-1}(\bar{u}^2)$. It remains to show that:

(i) for any of the $\alpha^{2*}_{t^k}$ there is a positive measure of values of δ for which

$$v\left(\alpha^{2*}_{t^k}\left(\frac{p^*_{t^k} + \delta}{p^*_{t^k}}\right) + \left(1-\alpha^{2*}_{t^k}\right)(1+r)\right) < \bar{u}^2$$

holds;

(ii) the sequence $\alpha^{2*}_{t^k}$ crosses $\frac{1}{2}$ from below a.s. in finite time.

These two statements are proven below. Once this value of $\alpha_{t^n}^{2*} \geq \frac{1}{2}$ and the corresponding value of $p_{t^n}^*$ are reached,

$$\alpha_{t^n}^{2*} = \frac{p_{t^n}^* - \theta_1\alpha_0}{1-\theta_1} \geq \frac{\frac{2\theta_1(1-\theta_1)\alpha_0 + (1-\theta_1)}{2} - \theta_1\alpha_0}{1-\theta_1} = \frac{1}{2}$$

obtains and from the next period, $\bar{t}(\tilde{\omega})$ such that:

$$\bar{t}(\tilde{\omega}) = \min\left\{ t > t^n \mid u\left(\alpha_{t^n}^{*2}\left(\frac{p_{t^n}^* + \delta_t}{p_{t^n}^*}\right) + \left(1-\alpha_{t^n}^{*2}\right)(1+r)\right) < \bar{u}^2 \right\}$$

$\alpha_{\bar{t}(\tilde{\omega})}^{2*} = \alpha_t^{2*} = 0$ for all $t > \bar{t}(\tilde{\omega})$. Since $\bar{t}(\tilde{\omega})$ is a.s. finite, this completes the proof of the proposition.

Proof of statements (i) and (ii):

(i) Note that

$$\alpha_{t^k}^{*2}\left(\frac{p_{t^k}^* + \underline{\delta}}{p_{t^k}^*}\right) + \left(1-\alpha_{t^k}^{*2}\right)(1+r) < \alpha_0\left(\frac{p_0 + \underline{\delta}}{p_0}\right) + (1-\alpha_0)(1+r),$$

is equivalent to

$$\begin{aligned} & r\left(\alpha_0 - \alpha_{t^k}^{*2}\right) + \frac{\underline{\delta}\alpha_{t^k}^{*2}}{p_{t^k}^*} - \underline{\delta} \\ = & \left(\alpha_0 - \alpha_{t^k}^{*2}\right)\left[\frac{r - \underline{\delta}\theta_1}{p_{t^k}^*}\right] < 0, \end{aligned}$$

which is satisfied for any k, since $\alpha_0 < \alpha_{t^k}^{*2}$ and $r > \underline{\delta} > \underline{\delta}\theta_1$ hold.

(ii) $\alpha_{t^n}^{*2} \geq \frac{1}{2}$ is equivalent to

$$p_{t^n}^* \geq \frac{1-\theta_1+2\theta_1\alpha_0}{2} = \theta_1\alpha_0 + \frac{1}{2}(1-\theta_1)$$

Observe that:

$$p_{t^k}^* - p_{t^{k-1}}^* = \frac{p_{t^{k-1}}^*(1-\theta_1)}{p_{t^{k-1}}^* - \theta_1\alpha_0}\left[\varpi - (1+r) + \left(r - \frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}}\right], \qquad (6)$$

Note that

$$\frac{p_{t^{k-1}}^*(1-\theta_1)}{p_{t^{k-1}}^* - \theta_1\alpha_0} > 1-\theta_1,$$

whereas

$$\frac{1-\theta_1+2\theta_1\alpha_0}{2} - \theta_1\alpha_0 = \frac{1-\theta_1}{2}$$

is the least amount by which p_{t^k} should grow to obtain a value higher than $\frac{1-\theta_1+2\theta_1\alpha_0}{2}$.

However,

$$\begin{aligned} & \varpi - (1+r) + \left(r - \frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}} - \left[\varpi - (1-\alpha_0)(1+r) - \alpha_0 - \delta_{t'}\right] \\ = & \varpi - \left(1-\alpha_{t^{k-1}}^{2*}\right)(1+r) - \left(\frac{p_{t^{k-1}}^* + \delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}}^{2*} - \left[\varpi - (1-\alpha_0)(1+r) - \alpha_0 - \delta_{t'}\right] \end{aligned}$$

$$= \quad r\left(\alpha_{t'}^{2*} - \alpha_0\right) + \frac{\delta_{t^k}}{p_{t^{k-1}}^*}\alpha_{t^{k-1}}^{2*} - \delta_{t'}$$

and by the choice of t',

$$\varpi - (1-\alpha_0)(1+r) - \alpha_0 - \delta_{t'} > 0$$

holds. Hence,

$$\varpi - (1+r) + \left(r - \frac{\delta_{t^k}}{p_{t^{k-1}}^*}\right)\alpha_{t^{k-1}} > r\left(\alpha_{t'}^{2*} - \alpha_0\right) > 0$$

if

$$\frac{\delta_{t^k}}{p_{t^{k-1}}^*}\alpha_{t^{k-1}}^{2*} - \delta_{t'} \geq 0.$$

Choose z such that

$$\frac{1}{2z} = r\left(\alpha_{t'}^{2*} - \alpha_0\right).$$

Obviously, after at most z (not necessarily sequential) periods in which $\delta_{t^k} \leq \frac{\delta_{t'}}{\alpha_{t^{k-1}}^{2*}} p_{t^{k-1}}^*$ obtains, $p_{t^k}^* - p_0 > \frac{1-\theta_1}{2}$ holds. Hence, $p_{t^k}^* \geq \frac{1-\theta_1+2\theta_1\alpha_0}{2}$.

Let t^z denote the last z^{th} period in which $\delta_{t^k} \leq \delta_{t'}$ obtains. It follows that there exists a t^n such that

$$t^n = \min\left\{t^k \leq t^z \mid p_{t^k}^* - p_0 \geq \frac{1-\theta_1}{2}\right\}$$

and, therefore,

$$\begin{aligned} p_{t^n}^* &\geq \frac{1-\theta_1+2\theta_1\alpha_0}{2} \\ \alpha_{t^n}^{2*} &\geq \frac{1}{2} \end{aligned}$$

obtains. On the other hand, since $\frac{p_{t^{k-1}}^*}{\alpha_{t^{k-1}}^{2*}} > 1$, the probability that the number of periods in which $\delta_{t^k} \leq \frac{\delta_{t'}}{\alpha_{t^{k-1}}^{2*}} p_{t^{k-1}}^*$ obtains is less than z is 0 on the set of sample paths of dividend realizations. Hence, t^z and, therefore, also t^n are a.s. finite.■

Part 3

As shown above, for $u\left(1+\underline{\delta}+(1-\alpha_0)r\right) < \bar{u}^2$ either $\alpha_t^{2*} = 0$ or $\alpha_t^{2*} = 1$ obtains a.s. in finite time $\bar{t}$.

Case 1: $\alpha_{\bar{t}}^{2*} = 0$, $p_{\bar{t}}^* = \theta_1\alpha_0$.

$$\begin{aligned} \alpha_{\bar{t}+1}^{2*} &= \arg\max_{\alpha\in[0;1]} U_{t+1}^2(\alpha) = \arg\max_{\alpha\in[0;1]} s\left((p_{\bar{t}+1};\alpha);(\theta_1\alpha_0;0)\right)\left[u(1+r) - \bar{u}^2\right] = 1 \\ p_{\bar{t}+1}^* &= 1 - \theta_1(1-\alpha_0) \end{aligned}$$

Case 2: $\alpha_{\bar{t}}^{2*} = 1$, $p_{\bar{t}}^{*} = 1 - \theta_1 (1 - \alpha_0)$ and $\delta_{\bar{t}+1} < \hat{\delta}$.[21]

$$\alpha_{\bar{t}+1}^{2*} = \arg\max_{\alpha \in [0;1]} U_{\bar{t}+1}^{2} (\alpha) = \left[v_{\bar{t}+1} (1) - \bar{u}^2 \right] s \left(\left(p_{\bar{t}+1}; \alpha \right) ; \left(p_{\bar{t}}^{*}; 1 \right) \right) = 0$$

since

$$v_{\bar{t}+1} (1) \leq u \left(1 + \frac{\delta_{\bar{t}+1}}{1 - \theta_1 (1 - \alpha_0)} \right) - \bar{u}^2 < 0$$

and

$$p_{\bar{t}+1}^{*} = \theta_1 \alpha_0.$$

Case 3: $\alpha_{\bar{t}}^{2*} = 1$, $p_{\bar{t}}^{*} = 1 - \theta_1 (1 - \alpha_0)$ and $\delta_{\bar{t}+1} > \hat{\delta}$.

$$\begin{aligned} \alpha_{\bar{t}+1}^{2*} &= \arg\max_{\alpha \in [0;1]} U_{t+1}^{2} (\alpha) = \arg\max_{\alpha \in [0;1]} s \left(\left(p_{\bar{t}+1} = p_{\bar{t}}^{*}; \alpha \right) ; \left(p_{\bar{t}}^{*}; 1 \right) \right) \cdot \\ \cdot \left[u \left(1 + \frac{\delta_{\bar{t}+1}}{1 - \theta_1 (1 - \alpha_0)} \right) - \bar{u}^2 \right] &= 1 \\ p_{\bar{t}+1}^{*} &= 1 - \theta_1 (1 - \alpha_0), \end{aligned}$$

since

$$v_{\bar{t}+1} (1) = u \left(1 + \frac{\delta_{\bar{t}+1}}{1 - \theta_1 (1 - \alpha_0)} \right) > \bar{u}^2$$

Hence, p_t^{*} follows a Markov process with a transition matrix:

$$\bar{P} = \left(\begin{array}{|c|c|c|} \hline & p_{\bar{t}+1}^{*} = 1 - \theta_1 (1 - \alpha_0) & p_{\bar{t}+1}^{*} = \theta_1 \alpha_0 \\ \hline p_t^{*} = 1 - \theta_1 (1 - \alpha_0) & q & 1 - q \\ \hline p_t^{*} = \theta_1 \alpha_0 & 1 & 0 \\ \hline \end{array} \right),$$

where $q = \Pr \left\{ \delta \geq \hat{\delta} \right\}$. The invariant probabilities of the states h and l (as defined in the statement of the proposition) are easily computed to be

$$\begin{aligned} \bar{\pi}_h &= \frac{1}{2 - q} \\ \bar{\pi}_l &= \frac{1 - q}{2 - q}. \blacksquare \end{aligned}$$

Part 4

The proof of part 3 shows that a cycle with two states h and l emerges after a finite number of periods $\bar{t}$. Cases 1 and 2 considered in the proof of part 3 are still valid. It remains to consider

Case 3: $p_{\bar{t}}^{*} = p_h$, $\alpha_{\bar{t}}^{2*} = 1$ and $\delta_{\bar{t}+1} \geq \hat{\delta}$.

$$\begin{aligned} \alpha_{\bar{t}+1}^{2*} &= \arg\max_{\alpha \in [0;1]} U_{\bar{t}+1}^{2} (\alpha) = \arg\max_{\alpha \in [0;1]} s \left(\left(p_{\bar{t}+1}^{*}; \alpha \right) ; \left(p_{\bar{t}}^{*}; 1 \right) \right) \cdot \\ \cdot \left[v_{\bar{t}+1} (1) - \bar{u}^2 \right] &= 0 \end{aligned}$$

[21] $\hat{\delta} \in \left[\underline{\delta}; \bar{\delta} \right]$ as defined in the statement of the proposition exists if

$$\underline{\delta} \theta_1 < r,$$

which is always satisfied under the assumptions made.

$$p_{\bar{t}+1}^{*} = \theta_1 \alpha_0,$$

since

$$v_{\bar{t}+1}(1) = u\left(\frac{p_{\bar{t}+1} + \delta_{\bar{t}+1}}{1-\theta_1(1-\alpha_0)}\right) < \bar{u}^2 \text{ for any } p_{\bar{t}+1} \leq 1-\theta_1(1-\alpha_0). \blacksquare$$

Proof of proposition 2:

Part 1

Since

$$U_t^2(\alpha_0) = \sum_{\tau=1}^{t}\left[v_\tau(\alpha_0) - \bar{u}^2\right],$$

behaves as a random walk on $\mathbb{R}$ as long as $\alpha_t^{2*} = \alpha_0$. It has a positive expected increment, since

$$E\left[v_t(\alpha_0)\right] = \mu(\alpha_0 \mid p_0) > \bar{u}^2.$$

According to theorem 9.5.1 in Main and Tweedie (1996, p. 228) such random walks are transient, hence the expected time until their first return to 0 is infinite.$\blacksquare$

Part 2

If $\bar{u}^2 \in (\mu(\alpha_0 \mid p_0); u(1+r))$, define the process

$$\tilde{U}_t^2(\alpha_0) = \left\{\begin{array}{ll} U_t^2(\alpha_0), & \text{if } U_t^2(\alpha_0) \geq 0 \\ 0, & \text{else} \end{array}\right\}.$$

The assumption

$$\mu(\alpha_0 \mid p_0) < \bar{u}^2$$

implies that $\tilde{U}_t^2(\alpha_0)$ is a random walk on $\mathbb{R}_0^+$ with negative expected increments. For such random walks all compact sets are regular, see proposition 11.4.1 in Meyn and Tweedie (1996, p. 278), it follows that

$$\tilde{U}_t^2(\alpha_0) = 0$$

obtains in finite time with probability 1. Since the distribution Q is continuous, it follows that

$$U_t^2(\alpha_0) < 0$$

obtains a. s. in finite time.

Once $U_t^2(\alpha_0) < 0$ holds, apply the proof of part 3 of proposition 1 to show that $\alpha_{\bar{t}}^{2*} = 0$ or $\alpha_{\bar{t}}^{2*} = 1$ obtains a.s. in finite time. This result can be applied, since in period $\bar{t}(\tilde{\omega})$ (as constructed in the proof of part 2 of proposition 1)[22], $V_{\bar{t}(\tilde{\omega})}^2\left(\alpha_{t^k}^{2*}\right) = 0$ for all $t^k \leq t^{n-1}$, whereas

[22] In the case of long memory, however, the time periods t^k will not denote the *subsequent* periods in which the dividend realization is lower than $\tilde{\delta}$, but those periods in which $\delta_{t^k} < \tilde{\delta}$ and

$$U_{t^k}^2\left(\alpha_{t^{k-1}}^{2*}\right) + u\left(\alpha_{t^{k-1}}^{2*}(1+\delta_{t^k}) + \left(1-\alpha_{t^{k-1}}^{2*}\right)(1+r)\right) - \bar{u}^2 < 0,$$

$V^2_{\bar{t}(\tilde{\omega})}\left(\alpha^{2*}_{t^n}\right) \leq 0$. Since

$$\bar{u}^2 < u\left(1+r\right) < \mu\left(1 \mid 1-\theta_1\left(1-\alpha_0\right)\right).$$

$V^2_t\left(\alpha^{2*}_{\bar{t}}\right)$ and $U^2_t\left(\alpha^{2*}_{\bar{t}}\right)$ behave as random walks with positive expected increments and remain positive infinitely long in expectations. Hence, the expected time during which the investors of type 2 hold $\alpha = 1$ or $\alpha = 0$ is infinity.

Furthermore, for any portfolio α:

$$\begin{aligned} U^2_t\left(\alpha\right) &= s\left(\left(p_t;\alpha\right);\left(\left(1-\theta_1\left(1-\alpha_0\right);1\right)\right)\right)V^2_t\left(1\right) + s\left(\left(p_t;\alpha\right);\left(\left(\theta_1\alpha_0;0\right)\right)\right)V^2_t\left(0\right) \\ &\quad + s\left(\left(p_t;\alpha\right);\left(\left(p^*_{t^n};\alpha^{2*}_{t^n}\right)\right)\right)V^2_t\left(\alpha^{2*}_{t^n}\right), \end{aligned}$$

Obviously, if exactly one of the numbers $V^2_t\left(1\right)$, $V^2_t\left(0\right)$ or $V^2_t\left(\alpha^{2*}_{t^n}\right)$ is positive, then the corresponding act is chosen in the next period of time. The case-based decision rule precludes the case that two of these numbers are positive simultaneously at some t. This is shown in the following lemma:

Lemma 7 *Type 2 abandons a portfolio α only in periods $\tilde{t}$ such that $V^2_{\tilde{t}}\left(\alpha\right) < 0$.*

Proof of lemma 7

In the proof of proposition 1, it has already been shown that the statement of the lemma is true up to time $\bar{t}$ such that $\bar{t} = \min\left\{t \mid U^2_t\left(\alpha_0\right) < 0\right\}$.

To argue by induction, suppose that the statement holds up to a period $t-1$ and consider period t. Let $\alpha^{2*}_\tau \in \left\{\alpha_1 ... \alpha_l\right\}$ with $\alpha_l = \alpha_t$ and define $p^*\left(\alpha_i\right)$ as:

$$p^*\left(\alpha_i\right) =: \alpha_0\theta_1 + \left(1-\theta_1\right)\alpha_i \text{ for } i \in \left\{1...l\right\}.$$

Let $V^2_t\left(\alpha_l\right) \geq 0$. Then,

$$\begin{aligned} U^2_t\left(\alpha_l\right) &= \sum_{\substack{i=1 \\ i\neq l}}^{l} V^2_t\left(\alpha_i\right) s\left(\left(\alpha_i;p^*\left(\alpha_i\right)\right);\left(\alpha_l;p^*\left(\alpha_l\right)\right)\right) + V^2_t\left(\alpha_l\right) \geq \\ &\geq \sum_{\substack{i=1 \\ i\neq l}}^{l} V^2_{\bar{t}''}\left(\alpha_i\right) s\left(\left(\alpha_i;p^*\left(\alpha_i\right)\right);\left(\alpha_l;p^*\left(\alpha_l\right)\right)\right) + V^2_{\bar{t}''}\left(\alpha_l\right), \end{aligned}$$

---

whereas $U^2_{t^k-1}\left(\alpha^{2*}_{t^{k-1}}\right) \geq 0$ holds. The portfolio $\alpha^{2*}_{t^k}$ (and, hence, the price $p^*_{t^k}$) are then chosen in such a way that

$$U^2_{t^k}\left(\alpha^{2*}_{t^{k-1}}\right) + u\left(\frac{p^*_{t^k}+\delta_{t^k}}{p^*_{t^{k-1}}}\alpha^{2*}_{t^{k-1}} + \left(1-\alpha^{2*}_{t^{k-1}}\right)\left(1+r\right)\right) - \bar{u}^2 = 0$$

Hence, $U^2_{t^k}\left(\alpha\right) = 0$ for each $\alpha \in [0;1]$ and, therefore, the choices till time t^k do not influence the evaluation of the available portfolios.

where $\bar{t}''-1=\max\left\{\tau \mid \alpha_{\tau}^{2*}\neq l\right\}$. The inequality follows from the fact that $V_{\bar{t}''}^{2}\left(\alpha_{l}\right)\leq 0$, since either act α_{l} has been chosen for the first time at $\bar{t}''$ and, therefore, $V_{\bar{t}''}^{2}\left(\alpha_{l}\right)=0$ or α_{l} has been abandoned for the last time at some time $t''+1<\bar{t}''+1$ and then

$$V_{\bar{t}''}^{2}\left(\alpha_{l}\right)=V_{t''}^{2}\left(\alpha_{l}\right)<0$$

must hold. Since no acts different from α_{l} have been chosen after period $\bar{t}''$,

$$V_{\bar{t}''}^{2}\left(\alpha_{i}\right)=V_{t}^{2}\left(\alpha_{i}\right)$$

holds for $i\in\{1...l-1\}$.

Furthermore, since $\alpha_{\bar{t}''+1}^{2*}=\alpha_{l}$, for all $\alpha\in[0;1]$:

$$U_{\bar{t}''}^{2}\left(\alpha_{l}\right)=\sum_{i=1}^{l-1}V_{\bar{t}''}^{2}\left(\alpha_{i}\right)s\left(\left(\alpha_{i};p^{*}\left(\alpha_{i}\right)\right);\left(\alpha_{l};p^{*}\left(\alpha_{l}\right)\right)\right)+V_{\bar{t}''}^{2}\left(\alpha_{l}\right)\geq U_{\bar{t}''}^{2}\left(\alpha\right)$$

holds. But then

$$\begin{aligned}
U_{t}^{2}\left(\alpha_{l}\right)-U_{t}^{2}\left(\alpha\right) &= \sum_{i=1}^{l-1}V_{t}^{2}\left(\alpha_{i}\right)\left[s\left(\left(\alpha_{i};p^{*}\left(\alpha_{i}\right)\right);\left(\alpha_{l};p^{*}\left(\alpha_{l}\right)\right)\right)-s\left(\left(\alpha_{i};p^{*}\left(\alpha_{i}\right)\right);\left(\alpha;p^{*}\left(\alpha_{l}\right)\right)\right)\right] \\
&\quad +V_{t}^{2}\left(\alpha_{l}\right)\left(1-s\left(\left(\alpha_{l};p^{*}\left(\alpha_{l}\right)\right);\left(\alpha;p^{*}\left(\alpha_{l}\right)\right)\right)\right) \\
&\geq \sum_{i=1}^{l-1}V_{\bar{t}''}^{2}\left(\alpha_{i}\right)\left[s\left(\left(\alpha_{i};p^{*}\left(\alpha_{i}\right)\right);\left(\alpha_{l};p^{*}\left(\alpha_{l}\right)\right)\right)-s\left(\left(\alpha_{i};p^{*}\left(\alpha_{i}\right)\right);\left(\alpha;p^{*}\left(\alpha_{l}\right)\right)\right)\right] \\
&\quad +V_{\bar{t}''}^{2}\left(\alpha_{l}\right)\left(1-s\left(\left(\alpha_{l};p^{*}\left(\alpha_{l}\right)\right);\left(\alpha;p^{*}\left(\alpha_{l}\right)\right)\right)\right) \\
&= U_{\bar{t}''}^{2}\left(\alpha_{l}\right)-U_{\bar{t}''}\left(\alpha\right)\geq 0.
\end{aligned}$$

Hence, $\alpha_{t+1}^{2*}=\alpha_{l}$ if $V_{t}^{2}\left(\alpha_{l}\right)\geq 0$ and hence, an act α_{l} can be only abandoned in a period $\tilde{t}$ such that $V_{\tilde{t}}^{2}\left(\alpha_{l}\right)<0$ holds.■

Hence, at most one of the numbers $V_{t}^{2}(1)$, $V_{t}^{2}(0)$ and $V_{t}^{2}\left(\alpha_{t^{n}}^{2*}\right)$ can be positive in which case the act with the positive V_{t}^{2} will be chosen. If all of them are negative, then the concavity of $s(\cdot;\cdot)$ implies that $U_{t}^{2}(\alpha)$ is a convex function of α. Therefore, $\alpha_{t}^{2*}\in\{0;1\}$.

On all paths of dividend realizations, on which $U_{t}^{2}(1)\geq U_{t}^{2}(0)$ holds for all $t\geq\bar{t}(\tilde{\omega})$, $\alpha_{t}^{2*}=1$ has a frequency of 1. On those paths, on which $U_{t}^{2}(1)<U_{t}^{2}(0)$ obtains for some $T\geq\bar{t}(\tilde{\omega})$, $\alpha_{T+1}^{2*}=0$. Then,

$$u(1+r)>\bar{u}^{2}$$

implies that $U_{t}^{2}(1)<U_{t}^{2}(0)$ and $\alpha_{t}^{2*}=0$ hold for all $t>T$. Hence, the limit frequency of $\alpha_{t}^{2*}=0$ is 1.■

Part 3

If

$$\bar{u}^2 \in (u(1+r); \mu(1 \mid 1-\theta_1(1-\alpha_0))), \tag{7}$$

either $\alpha_{\bar{t}}^{2*} = 1$ or $\alpha_{\bar{t}}^{2*} = 0$ obtains in finite time, as shown in part 2 of this proof. If $\alpha_{\bar{t}}^{2*} = 0$ then (7) implies that $U_t^2(0) < U_t^2(1)$ obtains in finite time. If $\alpha_{\bar{t}}^{2*} = 1$, then $U_t^2(1)$ behaves like a random walk with positive expected increments and $\alpha_t^{2*} = 1$ holds infinitely long in expectation.

As in part 2, the concavity of $s(\cdot;\cdot)$ implies that $\alpha_t^{2*} \in \{0;1\}$ for all $t \geq \bar{t}$.

If $U_t^2(0) > U_t^2(1)$ obtains at some time t and, therefore $\alpha_t^{2*} = 0$, the argument above shows that $U_{t'}^2(0) < U_{t'}^2(1)$ will obtain again for some finite $t' > t$, hence $\alpha_{t'}^{2*} = 1$. Since

$$\Pr\left\{U_t^2(1) < U_t^2(0) \text{ for some } t > t_0 \mid U_{t_0}^2(1) > U_{t_0}^2(0)\right\} < 1$$

and these events are independent, the probability that such events occur infinitely often is 0. Hence, in the limit, a.s. $\alpha_t^{2*} = 1$ holds with frequency 1.■

Part 4

Now, let $\bar{u}^2 > \mu(1 \mid 1-\theta_1(1-\alpha_0))$. Let s denote:

$$s = s((p = 1-\theta_1(1-\alpha_0); \alpha = 1); (p = \theta_1\alpha_0; \alpha = 0)) \in [0;1).$$

As above, it can be shown that there is a finite $\bar{t}$ such that $\alpha_t^{2*} \in \{0;1\}$ for all $t \geq \bar{t}$. Denote by ε_t the following Markov process:

$$\begin{aligned}
\varepsilon_0 &= \frac{V_t^2(\alpha_{t^n}^{2*})\left[s((p_h;1);((p_{t^n}^*;\alpha_{t^n}^{2*}))) - s((p_l;0);((p_{t^n}^*;\alpha_{t^n}^{2*})))\right]}{1-s} =: \bar{V} \\
\varepsilon_t &= \left\{\begin{array}{ll}
\varepsilon_{t-1} + u\left(1+\frac{\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(1+\frac{\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2 \geq 0 \\
\varepsilon_{t-1} + u\left(\frac{\theta_1\alpha_0+\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2, & \text{if } \varepsilon_{t-1} \geq 0 \text{ and } \varepsilon_{t-1} + u\left(\frac{\theta_1\alpha_0+\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2 < 0 \\
\varepsilon_{t-1} + u(1+r) - \bar{u}^2, & \text{if } \varepsilon_{t-1} < 0.
\end{array}\right\}
\end{aligned}$$

for $t \geq 1$.

It is easily seen that

$$\varepsilon_t(1-s) = U_t^2(1) - U_t^2(0)$$

Since

$$U_t^2(1) - U_t^2(0) \in \left[\left[u\left(\frac{\theta_1\alpha_0 + \underline{\delta}}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2\right](1-s); +\infty\right),$$

ε_t evolves on

$$\Psi' = \left[u\left(\frac{\theta_1\alpha_0 + \underline{\delta}}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2; +\infty\right),$$

Denote by P the transition probability kernel of ε_t. The idea of the proof consists in showing that ε_t is a stationary process with an invariant probability measure π, as defined in the statement of the proposition. If $\varepsilon_t \geq 0$, $\alpha_t^{2*} = 1$, if $\varepsilon_t < 0$, $\alpha_t^{2*} = 0$. Hence the limit frequencies π_h and π_l of $\alpha_t^{2*} = 1$ and $\alpha_t^{2*} = 0$ coincide with

$$\pi\left[0; +\infty\right)$$

and

$$\pi\left[u\left(\frac{\theta_1\alpha_0 + \underline{\delta}}{1 - \theta_1\left(1 - \alpha_0\right)}\right) - \bar{u}^2; 0\right),$$

respectively.

The following lemmas and proposition prove that ε_t is a positive recurrent Harris chain and, hence, has an invariant probability distribution by using the following intermediate steps:

1. Lemma 8 identifies a small set G, which is also a petite set.

2. Lemma 9 uses the small set G to show that ε_t is ψ-irreducible, where ψ denotes the Lebesgue measure on G and is 0, elsewhere.

3. Proposition 10 reproduced from Meyn and Tweedie (1996) states that ψ-irreducibility implies positive recurrence if G is reached in finite expected time from any value of ε.

In a last step, I demonstrate that the condition of 10 is satisfied for the chain ε and compute the invariant probability distribution.

Denote by G the interval $[0; \bar{u}^2 - u\left(1 + r\right)]$. The following lemma shows that the set G is a small set, i.e. that there exists a measure ν on the set

$$\Psi' = \left[u\left(\frac{\theta_1\alpha_0 + \underline{\delta}}{1 - \theta_1\left(1 - \alpha_0\right)}\right) - \bar{u}^2; +\infty\right)$$

such that

$$P^K\left(\varepsilon; F\right) \geq \nu\left(F\right)$$

for any set $F \in \Psi'$ and any $\varepsilon \in G$, where $P^K\left(\varepsilon; F\right)$ denotes the probability to reach a set F starting from ε in K steps, see Meyn and Tweedie (1996, p. 111).

Lemma 8 *The set* $G = [0; \bar{u}^2 - u\left(1 + r\right)]$ *is small.*

Proof of lemma 8:

The assumption about the probability distribution of δ and the continuity of the utility function

$u(\cdot)$ imply that the net utility realizations

$$\tilde{u} = u\left(1 + \frac{\delta_t}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2$$

of $\alpha_t^{2*} = 1$ (as long as $U_t^2(1) \geq U_t^2(0)$) are distributed according to a probability distribution Q', which has an absolutely continuous part with respect to the Lebesgue measure on the real numbers. Moreover, there is a number ζ', such that the density of $\tilde{u}$, g' satisfies

$$g'(\tilde{u}) \geq \phi' > 0 \text{ for } \tilde{u} \in (-\zeta'; \zeta')$$

for some $\zeta' \in (0; \bar{u}^2 - u(1+r))$ and for some ϕ'.

Divide the set G into K sets, $G_1 ... G_K$ with length less than $\frac{\zeta'}{2}$. Fix an initial $\varepsilon \in G_i$ and suppose that $F \subset G_j$. Now, for each $0 < \xi < \frac{\zeta'}{2}$, denote by $P_{\xi-}$ and P_{ξ^+}

$$P_{\xi-} = \Pr\left\{\varepsilon_{t+1} \in \left(\varepsilon_t + \frac{\zeta'}{2} - \xi; \varepsilon_t + \frac{\zeta'}{2}\right)\right\}$$
$$P_{\xi^+} = \Pr\left\{\varepsilon_{t+1} \in \left(\varepsilon_t - \frac{\zeta'}{2}; \varepsilon_t - \frac{\zeta'}{2} + \xi\right)\right\}$$

The assumptions on Q imply

$$\begin{aligned} P_{\xi^+} &\geq \phi'\xi > 0 \\ P_{\xi^-} &\geq \phi'\xi > 0. \end{aligned}$$

Now choose ξ such that $\xi(K-1) \leq \frac{\zeta'}{2}$ holds. Then,

$$\Pr\{G_j - \varepsilon_{K-1} \leq \zeta' \mid \varepsilon_0 = \varepsilon\} \geq [\phi'\xi]^{K-1},$$

where $G_j - \varepsilon_{K-1}$ is the largest distance between a point in G_j and ε_{K-1}. Hence,

$$P(\varepsilon_{K-1}; F \mid G_j - \varepsilon_{K-1} \leq \zeta') = Q'(F - \varepsilon_{K-1}) = \int_0^{F-\varepsilon_{t+K-1}} g'(\tilde{u})\, d\tilde{u} \geq \phi'\lambda(F),$$

where λ denote the Lebesgue measure. Hence,

$$P^K(\varepsilon; F) \geq [\phi'\xi]^{K-1} \phi'\lambda(F) =: \nu(F),$$

and ν is absolutely continuous w.r.t. λ on G. If $F \subset G$ but $F \nsubseteq G_j$ for all G_j, then:

$$P^K(\varepsilon; F) = \sum_{i=1}^{K} P^K(\varepsilon; F_i) \geq [\phi'\xi]^{K-1} \phi' \sum_{i=1}^{K} \lambda(F_i) = [\phi'\xi]^{K-1} \phi'\lambda(F),$$

where $\cup_{i=1}^K F_i = F$ and $F_i \subset G_i$, i.e. F_i is a partition of F into sets each of which is a (possibly empty) subset of some G_i. Since for all F such that $F \cap G = \emptyset$,

$$P^K(\varepsilon; F) \geq 0,$$

G is a small set and the measure $\nu(F)$ is defined as

$$\nu(F) = [\phi'\xi]^{K-1} \phi'\lambda(F), F \subset G$$

$$\nu(F) = 0\text{, else.}$$

According to proposition 5.5.3 in Meyn and Tweedie (1996, p. 127), G is also a petite set.■

The next Lemma demonstrates that the Markov chain defined by ε_t is φ-irreducible. It defines a measure φ, with $\varphi(F) > 0$ only if $F \subset \Psi'$ satisfies

$$P^k(\varepsilon; F) > 0$$

for some $k \in \mathbb{N}$ and all $\varepsilon \in \Psi'$, see Meyn and Tweedie (1996, p. 91).

Lemma 9 *Let φ be defined as the Lebesgue measure on the set G and be 0 elsewhere. Then the Markov chain ε is φ-irreducible.*

Proof of lemma 9:

Obviously, φ assigns a positive probability only to subsets of the interval G. Since it has been shown that starting from any $\varepsilon \in G$, $P^K(\varepsilon; F) > 0$ for all $F \subset G$, it remains to demonstrate that for each $\varepsilon \notin G$, $P^k(\varepsilon; G) > 0$ for some $k \in \mathbb{N}$. First let $\varepsilon < 0$. Then $\varepsilon_{t+1} = \varepsilon_t + \bar{u}^2 - u(1+r)$ as long as $\varepsilon_t < 0$. Hence, for $k = \min\{t \mid \varepsilon_t \geq 0\}$, $\varepsilon_k \in G$. If $\varepsilon > \bar{u}^2 - u(1+r)$,

$$\Pr\left\{\tilde{u}_1 \in \left(-\frac{\zeta'}{2}; -\frac{\eta\zeta'}{2}\right) ... \tilde{u}_{\left\lceil 2\frac{\varepsilon_t}{\zeta'}\right\rceil} \in \left(-\frac{\zeta'}{2}; -\frac{\eta\zeta'}{2}\right)\right\} > 0$$

with $\left\lceil 2\frac{\varepsilon_t}{\zeta'}\right\rceil \frac{\eta\zeta'}{2} < \bar{u}^2 - u(1+r)$, (hence, $\eta < 1$) and, therefore,

$$P^{\left\lceil 2\frac{\varepsilon_t}{\zeta'}\right\rceil}\left(\varepsilon; \left[0; \left\lceil 2\frac{\varepsilon_t}{\zeta'}\right\rceil \frac{\eta\zeta'}{2}\right] \subset G\right) > 0$$

for all $\varepsilon \notin G$. Hence, the Markov chain is φ-irreducible.■

Since φ is finite, according to proposition 4.2.2 in Meyn and Tweedie (1996, p. 92), there exists a probability measure ψ on Ψ', such that $\psi(F \subset \Psi') = 0$ iff

$$\psi\left(\varepsilon \mid \sum_{n=1}^{\infty} P^n(\varepsilon; F) > 0\right) = 0.$$

ψ is absolutely continuous with respect to φ. Denote by $\mathcal{B}(\Psi')$ the Borel σ-algebra on Ψ'. Let

$$\mathcal{B}^+(\Psi') =: \{F \in \mathcal{B}(\Psi') \mid \psi(F) > 0\}.$$

Obviously, $G \in \mathcal{B}^+(\Psi')$.

We can now use part (ii) of theorem 10.4.10 in Meyn and Tweedie (1996, p. 254):

Proposition 10 *(Meyn and Tweedie, 1996, p. 254) Suppose that a Markov chain is ψ-irreducible. Let τ_G denote the first hitting time of the set G. The chain is positive recurrent, if for some petite set $G \in \mathcal{B}^+(\Psi')$*

$$\sup_{\varepsilon \in G} E_\varepsilon[\tau_G] < \infty. \tag{8}$$

Proof of proposition 10:

See Meyn and Tweedie (1996, p. 254).■

It has been shown that ε_t is ψ-irreducible and G is a petite set with $\psi(G) > 0$. It remains, to show (8). Consider $\tilde{\varepsilon}_t = \varepsilon_t$ if $\varepsilon_t \geq 0$ and $\tilde{\varepsilon}_t = 0$, else. $\tilde{\varepsilon}_t$ is a random walk on a half line with negative expected increments. By proposition 11.4.1 in Meyn and Tweedie (1996, p. 278) for such a random walk, all compact sets are regular. Since G is compact and $G \in \mathcal{B}^+(\Psi')$, it follows that for all $F \in \mathcal{B}^+(\Psi')$,

$$\sup_{\varepsilon \in G} E_\varepsilon [\tau_G] < \infty$$

and

$$\sup_{\varepsilon \in F} E_\varepsilon [\tau_G] < \infty \tag{9}$$

holds for the process $\tilde{\varepsilon}_t$.

Hence, on all paths on which $\varepsilon_t \geq 0$ holds for all t, ε_t coincides with $\tilde{\varepsilon}_t$ and G is reached in finite expected time. On those paths, on which $\varepsilon_t < 0$ holds for some t, the time needed to reach G is at most:

$$\frac{\bar{u}^2 - u\left(\frac{\theta_1 \alpha_0 + \underline{\delta}}{1 - \theta_1(1-\alpha_0)}\right)}{\bar{u}^2 - u(1+r)} < \infty.$$

Therefore, (8) is satisfied and ε_t is positive recurrent. Furthermore, (9) implies that ε_t is a positive recurrent Harris chain. Hence, by Theorem 10.0.1 in Meyn and Tweedie (1996, p. 238), there exists an invariant probability measure π.

It now remains to show that π_h and π_l as defined in the statement of the proposition are positive and satisfy:

$$\frac{\pi_h}{\pi_l} = \frac{\bar{u}^2 - u(1+r)}{\bar{u}^2 - \mu_a^r}.$$

According to the SLLN, if $\alpha_t^{2*} = 1$ holds in an infinite number of periods,

$$\lim_{t\to\infty} V_t^2(1) = -\infty$$

holds. Analogously, if $\alpha_t^{2*} = 0$ holds in an infinite number of periods,

$$\lim_{t\to\infty} V_t^2(b) = -\infty$$

obtains.

If, e.g. $\alpha_t^{2*} = 0$ holds only for a finite number of periods, say L

$$\lim_{t\to\infty}\left[U_t^2(1) - U_t^2(b)\right] = \lim_{t\to\infty}(1-s)V_t^2(1) - (1-s)V_t^2(0) = -\infty.$$

Hence, a.s. there exists a period $T > L$ such that $U_t^2(0) > U_t^2(1)$ for all $t > T$ and nevertheless $\alpha_t^{2*} = 1$, in contradiction to the case-based rule. Hence, a.s.

$$\begin{aligned}\lim_{t\to\infty} \left|C_t\left(\alpha_t^{2*} = 0\right)\right| &= \infty \\ \lim_{t\to\infty} \left|C_t\left(\alpha_t^{2*} = 1\right)\right| &= \infty.\end{aligned}$$

It can be shown that

$$U_t^2(1) - U_t^2(0) = (1-s)\,\varepsilon_t$$

remains a.s. bounded above. Obviously, $\alpha_t^{2*} = 1$ iff $\varepsilon_t \geq 0$. Suppose that there is a sequence of periods t', t''..., such that $\varepsilon_{t'}$, $\varepsilon_{t''}$... grows to infinity. In other words, suppose that for each $N > 0$, there is a k such that $\varepsilon_{t^n} > N$ for all $n > k$. Since

$$\lim_{t\to\infty} \left|C_t\left(\alpha_t^{2*} = 0\right)\right| = \infty,$$
$$\lim_{t\to\infty} \left|\{t \mid \varepsilon_t < 0\}\right| = \infty.$$

If $\varepsilon_{t^n} > N$, the time needed to reach $\varepsilon_t < 0$ is at least

$$\frac{N}{\bar{u}^2 - u\left(\frac{\theta_1\alpha_0 + \delta}{1-\theta_1(1-\alpha_0)}\right)},$$

which grows to infinity, as ε_{t^n} becomes very large. However, we know that

$$\tilde{N} = \sup_{\varepsilon\in G} E_\varepsilon\left[\tau_G\right] < \infty.$$

The LLN then implies that for each $\kappa > 0$, there is a.s. a period $\mathcal{K}$ with

$$\frac{\sum_{i=1}^n \tau_{G_i}}{n} \leq \tilde{N} + \kappa$$

for all $n \geq \mathcal{K}$. However, $\varepsilon_{t^n} \to \infty$ implies that there is a time $\mathcal{K}'$ such that $\tau_{G_i} > \tilde{N} + \kappa$ for all $i \geq \mathcal{K}'$. Hence, it is possible to choose n large enough, so that

$$\frac{\sum_{i=1}^n \tau_{G_i}}{n} > \tilde{N} + \kappa,$$

a contradiction. Hence, almost each sequence $\varepsilon_{t'}$, $\varepsilon_{t''}$... (with $\alpha_t^{2*} = 1$ at t', t''...) is bounded above and below.

At times at which $\alpha_t^{2*} = 0$, $\varepsilon_t \in \left[u\left(\frac{\theta_1\alpha_0+\delta}{1-\theta_1(1-\alpha_0)}\right) - \bar{u}^2; 0\right]$.

It follows that

$$\begin{aligned} & \lim_{t\to\infty} \frac{U_t^2(1)}{U_t^2(0)} \\ = \; & \lim_{t\to\infty} \frac{U_t^2(1)}{U_t^2(1) - (1-s)\,\varepsilon_t} \\ = \; & \lim_{t\to\infty} \frac{V_t^2(1) + sV_t^2(0) + V_t^2\left(\alpha_{t^n}^{2*}\right) s\left(\left(p_h; 1\right); \left(\left(p_{t^n}^*; \alpha_{t^n}^{2*}\right)\right)\right)}{sV_t^2(1) + V_t^2(0) + V_t^2\left(\alpha_{t^n}^{2*}\right) s\left(\left(p_h; 1\right); \left(\left(p_{t^n}^*; \alpha_{t^n}^{2*}\right)\right)\right)} = 1 \end{aligned}$$

a.s. holds. Since $V_t^2\left(\alpha_{t^n}^{2*}\right)$ is finite,

$$\lim_{t\to\infty}\frac{\left|C_t^2(1)\right|\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+s\left|C_t^2(0)\right|\frac{\sum_{\tau\in C_t^2(0)}\left(u(1+r)-\bar{u}^2\right)}{\left|C_t^2(0)\right|}}{s\left|C_t^2(1)\right|\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+\left|C_t^2(0)\right|\frac{\sum_{\tau\in C_t^2(0)}\left(u(1+r)-\bar{u}^2\right)}{\left|C_t^2(0)\right|}}=1. \tag{10}$$

Now define a function $\iota_h:\Psi'\to\{0;1\}$ with

$$\iota_h(x)=\left\{\begin{array}{ll}1\text{ if} & x\in[0;+\infty)\\ 0\text{ if} & x\in\left[u\left(\frac{\theta_1\alpha_0+\delta}{1-\theta_1(1-\alpha_0)}\right)-\bar{u}^3;0\right)\end{array}\right\}.$$

It is clear that $\iota_h\in L_1\left(\Psi';\mathcal{B}\left(\Psi'\right);\pi\right)$. Since ε is positive Harris recurrent, theorem 17.1.7 in Meyn and Tweedie (1996, p. 425) implies that

$$\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\iota_h\left(\varepsilon_\tau\right)=\int\iota_h d\pi$$

a.s. for any initial probability distribution. Hence,

$$\lim_{t\to\infty}\frac{1}{t}\sum_{\tau=1}^{t}\iota_h\left(\varepsilon_\tau\right)=\pi_h$$

obtains a.s. and the limit frequency of $\alpha_t^{2*}=1$ a.s. equals π_h. It follows that:

$$\lim_{t\to\infty}\frac{\left|C_t^2(1)\right|}{\left|C_t^2(0)\right|}=\frac{\pi_h}{\pi_l}$$

holds with probability 1, as well. Hence, (10) implies

$$\lim_{t\to\infty}\frac{\frac{\pi_h}{\pi_l}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+s\left(u(1+r)-\bar{u}^2\right)}{s\frac{\pi_h}{\pi_l}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}+\left(u(1+r)-\bar{u}^2\right)}=1$$

and

$$\frac{\pi_1}{\pi_0}\lim_{t\to\infty}\frac{\sum_{\tau\in C_t^2(1)}\left(v_\tau(1)-\bar{u}^2\right)}{\left|C_t^2(1)\right|}=\left(u(1+r)-\bar{u}^2\right),$$

a.s. holds. Hence, there exists a μ_1^r such that

$$\mu_1^r=\lim_{t\to\infty}\frac{\sum_{\tau\in C_t^2(1)}v_\tau(1)}{\left|C_t^2(1)\right|}$$

$$\frac{\pi_1}{\pi_0}=\frac{u(1+r)-\bar{u}^2}{\mu_1^r-\bar{u}^2}$$

obtain a.s.■

Proof of proposition 3

The proof of the proposition proceeds in two steps. First, I show that for each open set $A'\subset[0;1]$, $\alpha_t^{2*}\in A'$ holds in an infinite number of periods. This is an implication of the convexity of the similarity function and the negativity of net expected payoffs. Second, as in the proof of proposition 2, it is shown that $U_t^2(\alpha)-U_t^2(\alpha')$ remains bounded in the limit for all $\alpha,\alpha'\in[0;1]$.

This implies the result of the proposition.

Note that

$$\begin{aligned}\lim_{t\to\infty}\left|C_t^2(\alpha)\right| &= \infty \text{ for some } \alpha\in[0;1] \text{ and}\\ \lim_{t\to\infty}\left|C_t^2(\alpha')\right| &< \infty \text{ for all } \alpha'\neq\alpha\end{aligned}$$

cannot obtain on an equilibrium path, since for all $\alpha'\neq\alpha$

$$\mu(\alpha\mid p^*(\alpha))-\bar{u}<0$$

implies:

$$\lim_{t\to\infty}\left[U_t^2(\alpha)-U_t^2(\alpha')\right]=\lim_{t\to\infty}V_t^2(\alpha)\left[1-s\left((\alpha';p^*(\alpha));(\alpha;p^*(\alpha))\right)\right]\to-\infty \text{ a.s.},$$

contradicting the case-based rule.

Let, therefore

$$\begin{aligned}\lim_{t\to\infty}\left|C_t^2(\alpha')\right| &= \infty,\ \lim_{t\to\infty}\left|C_t^2(\alpha'')\right|=\infty \text{ for some } \alpha',\alpha''\in[0;1] \text{ and}\\ \lim_{t\to\infty}\left|C_t^2(\alpha)\right| &< \infty \text{ for all } \alpha\neq\alpha',\alpha''.\end{aligned}$$

Then, there exists a T such that $\alpha_t^2\in\{\alpha';\alpha''\}$ for all $t>T$. Denote the distinct acts chosen in periods $1...T$ by $\alpha_1...\alpha_l$. The cumulative utility of act α at $t>T$ for the investors of type 2 is given by:

$$\begin{aligned}U_t^2(\alpha) &= V_t^2(\alpha')\,s\left((\alpha';p^*(\alpha'));(\alpha;p^*(\alpha'))\right)+V_t^2(\alpha'')\,s\left((\alpha'';p^*(\alpha''));(\alpha;p^*(\alpha''))\right)\\ &\quad+\sum_{i=1}^{l}V_t^2(\alpha_i)\,s\left((\alpha;p^*(\alpha_i));(\alpha_i;p^*(\alpha_i))\right).\end{aligned}$$

By lemma 7, type 2 abandons a portfolio α only if $V_t^2(\alpha)<0$. Hence, $V_t^2(\alpha_i)<0$ holds for all $i=1...l$. Whereas $V_t^2(\alpha_i)$ are finite for all $i=1...l$,

$$\lim_{t\to\infty}V_t^2(\alpha')=\lim_{t\to\infty}V_t^2(\alpha'')=-\infty,$$

a.s.. Hence, for almost each path ω there exists a $t(\omega)$ such that $V_t^2(\alpha')<0$ and $V_t^2(\alpha'')<0$ for all $t\geq t(\omega)$.

Since the similarity function is convex, it follows that $U_t^2(\alpha)$ is strictly concave on the intervals:

$$\begin{gathered}\left((\alpha';p);\left(\min\left\{\min_{i\in\{1...l\}}\{\alpha_i\mid\alpha_i>\alpha'\};\alpha''\right\};p\right)\right)\\ \left(\left(\max\left\{0;\max_{i\in\{1...l\}}\{\alpha_i\mid\alpha_i<\alpha'\}\right\};p\right);(\alpha';p)\right),\end{gathered}$$

as well as on

$$\left((\alpha'';p);\left(\min\left\{\min_{i\in\{1...I\}}\{\alpha_i\mid\alpha_i>\alpha''\};1\right\};p\right)\right)$$

$$\left(\left(\max\left\{\alpha';\max_{i\in\{1\ldots I\}}\{\alpha_i \mid \alpha_i<\alpha''\}\right\};p\right);(\alpha'';p)\right)$$

for any $p \in [0;1]$. Hence, on almost each path ω, there exists a $T'(\omega)$ such that there are α''', $\alpha'^v \in [0;1]$ such that

$$\begin{aligned} U_t^2(\alpha''') &> U_t^2(\alpha') \\ U_t^2(\alpha'^v) &> U_t^2(\alpha'') \end{aligned}$$

hold at the equilibrium prices for all $t \geq T'(\omega)$ and still $\alpha_t^{2*} \in \{\alpha';\alpha''\}$. This contradicts the case-based rule. Clearly, the argument does not depend on the number of portfolios which are chosen infinitely often, as long as this number remains finite. Hence, an infinite (but countable) set of portfolios $A' \subset [0;1]$ must be chosen infinitely often by the investors of type 2.

Suppose now that A' does not contain an act out of $B_x(\epsilon)$ for some $x \in (0;1)$. By an argument similar to the above, we could find an element of $B_x(\epsilon)$, $\tilde{\alpha}$ which has been chosen only for a finite number of times and show that from some point of time $T''(\omega)$, the cumulative utility for the investors of type 2 of the portfolios in the interval

$$(\sup A' \backslash [x+\epsilon;1];\tilde{\alpha})$$

is a concave function for all $t \geq T''(\omega)$. Hence, for all

$$\begin{gathered} \alpha \in (\sup A' \backslash [x+\epsilon;1];\tilde{\alpha}) \\ U_t^2(\alpha) > U_t^2(\sup A' \backslash [x+\epsilon;1]). \end{gathered}$$

Since the similarity function is continuous, so is the cumulative utility function and therefore, there exists a portfolio $\alpha' \in A'$ which is chosen infinitely often by the investors of type 2 and the cumulative utility (for type 2) of which lies below the cumulative utility of α in each period $t \geq T''(\omega)$, a contradiction.

To complete the proof of the proposition, I now show that the difference between the cumulative utilities of any two portfolios:

$$U_t^2(\alpha) - U_t^2(\alpha') =: \tilde{\varepsilon}_t(\alpha;\alpha') \tag{11}$$

a.s. remains bounded over time. Since the expected mean payoffs of all acts are negative,

$$\lim_{t\to\infty} U_t^2(\alpha) = -\infty$$

a.s. for all acts $\alpha \in A$. This implies that

$$\lim_{t\to\infty}\frac{U_t^2(\alpha)}{U_t^2(\alpha')} = \lim_{t\to\infty}\frac{U_t^2(\alpha)}{U_t^2(\alpha)+\tilde{\varepsilon}_t(\alpha;\alpha')} = 1$$

holds on all paths on which $\tilde{\varepsilon}_t(\alpha;\alpha')$ remains bounded.

Hence, the proof of the following lemma completes the proof of proposition 3:

Lemma 11 *Define $\tilde{\varepsilon}_t(\alpha;\alpha')$ as in (11). On almost each path ω, $\tilde{\varepsilon}_t(\alpha;\alpha')$ is bounded.*

Proof of lemma 11

Consider first the portfolios in A' which are chosen infinitely often by the investors of type 2. For these acts, the proof that the difference between their cumulative utilities remains bounded with probability 1 is analogous to the argument used in the proof of proposition 2 on page 34.

It has been shown that there is no open subset of $[0;1]$ such that A' does not contain a portfolio out of this interval, hence for each $\epsilon > 0$, and $x \in (0;1)$, there is an $\alpha \in A' \cap B_x(\epsilon)$. Moreover, for all $\epsilon > 0$,

$$\lim_{t\to\infty} \frac{U_t^2(\tilde{\alpha})}{U_t^2(\alpha)} = 1,$$

where $\tilde{\alpha} \in A' \cap B_x(\epsilon)$ and $\alpha \in A'$, $\alpha \neq \tilde{\alpha}$. Since

$$\lim_{\epsilon\to 0} A' \cap B_x(\epsilon) = x$$

and $U_t^2(\tilde{\alpha})$ is continuous in $\tilde{\alpha}$, it follows that

$$\lim_{t\to\infty} \frac{U_t^2(x)}{U_t^2(\alpha)} = 1,$$

even if $x \notin A'$. This completes the proof of the proposition.■

Proof of proposition 4

The proof of proposition 3 has shown that if there is an open subset $\tilde{A} \subset [0;1]$ such that

$$\mu(\alpha \mid p^*(\alpha)) - \bar{u}^2 > 0$$

for all $\alpha \in \tilde{A}$, then the investors of type 2 will eventually choose a portfolio out of this set. The same proof further implies that the set of infinitely chosen acts cannot lie completely outside $\tilde{A}$. This means that $\lim_{t\to\infty} |C_t^2(\alpha)| = \infty$ for some $\alpha \in \tilde{A}$. Suppose, contrary to the statement of the proposition, that there are two distinct portfolios from $\tilde{A}$, $\alpha \neq \alpha'$, chosen with positive frequency by type 2. It is easy to show that this leads to a contradiction.

Indeed, consider the periods $z_{1\alpha}$, $z_{2\alpha}$,...$\in \mathbb{N}$ in which type 2 switches to act α and denote by $z_{1\alpha'}$, $z_{2\alpha'}$,...$\in \mathbb{N}$ the times, at which he switches to α'. The proof of lemma 7 shows that:

$$V_{z_{1\alpha}}^2(\alpha) > V_{z_{1\alpha'}}^2(\alpha) = V_{z_{2\alpha}}^2(\alpha) > V_{z_{2\alpha'}}^2(\alpha) = V_{z_{3\alpha}}^2(\alpha) > ...$$

But these inequalities imply that $V_t^2(\alpha)$, which (as long as α is chosen by type 2) is a random walk with positive expected increment $\mu(\alpha \mid p^*(\alpha)) - \bar{u}^2 > 0$, crosses each of the infinitely many boundaries $V^2_{z_{k\alpha}}(\alpha)$ from above. Since, however, there is a positive probability that a random walk with positive expected increment starting from a given point, never crosses a boundary lying below this point, see Grimmet and Stirzaker (1994, p. 144), and since the stopping times are independently distributed, it follows that the probability of infinitely many switches between α and α' is 0. Hence, only one of these two portfolios can be chosen by type 2 with positive frequency in the limit.

Alternatively, suppose that

$$\lim_{t\to\infty} \left|C_t^2(\alpha')\right| = \infty$$

with $\alpha' \in [0;1] \setminus \tilde{A}$. Then, a.s. for any $K > 0$, there exists a $\hat{T}(\omega)$ such that

$$U_t^2(\alpha) - U_t^2(\alpha') > K \text{ for all } t \geq \hat{T}(\omega)$$

and nevertheless, $\alpha_t^{2*} = \alpha'$ for some $t > \hat{T}(\omega)$, contradicting the case-based rule.■

Proof of corollary 5

Proposition 4 states that both types choose some portfolio α^{i*} ($i \in \{1;2\}$) with frequency one in the limit and the equilibrium price is constant at p^*. Moreover, $\mu(\alpha^{i*} \mid p^*) > \bar{u}$ holds. In this case,

$$\lim_{t\to\infty} U_t^i(\alpha) = +\infty$$

holds for all α. Moreover, for all α, α_1 and $\alpha_2 \in [0;1]$,

$$\begin{aligned}
\lim_{t\to\infty} \frac{U_t^i(a^{i*})}{U_t^i(\alpha)} &= \frac{1}{s((p^*;\alpha^{i*});(p^*;\alpha))} \geq 1 \\
\lim_{t\to\infty} \frac{U_t^i(a_1)}{U_t^i(a_2)} &= \frac{s((p^*;\alpha^{i*});(p^*;\alpha_1))}{s((p^*;\alpha^{i*});(p^*;\alpha_2))}.
\end{aligned}$$

If $\alpha^{i*} \in \{0;1\}$ for some $i \in \{1;2\}$, there will be no distinct α_1 and α_2 such that

$$\lim_{t\to\infty} \frac{U_t^i(\alpha_1)}{U_t^i(\alpha_2)} = 1$$

holds. Hence, the condition of preference for diversification is trivially satisfied. If, however, $\alpha^{i*} \in (0;1)$, then

$$s\left(\alpha^{i*};\alpha_1\right) = s\left(\alpha^{i*};\alpha_2\right),$$

iff

$$\|(p^*;\alpha^*) - (p^*;\alpha_1)\| = \|(p^*;\alpha^*) - (p^*;\alpha_2)\|. \tag{12}$$

Obviously, then for any α_1, α_2 satisfying (12) and $k \in \{1;2\}$,

$$\|(p^*;\alpha^*) - (p^*;\alpha_k)\| \geq \|(p^*;\alpha^*) - (p^*;\beta\alpha_1 + (1-\beta)\alpha_2)\|$$

for every $\beta \in [0;1]$. Hence,

$$s((p^*;\alpha^*);(p^*;\beta\alpha_1+(1-\beta)\alpha_2)) \geq s((p^*;\alpha^*);(p^*;\alpha_1)) = s((p^*;\alpha^*);(p^*;\alpha_2))$$

and, therefore $\beta\alpha_1+(1-\beta)\alpha_2$ is (weakly) preferred to α_k, $k \in \{1;2\}$ in the limit:

$$\lim_{t\to\infty} \frac{U_t^i(\beta\alpha_1+(1-\beta)\alpha_2)}{U_t^i(\alpha_k)} = \frac{s((p^*;\alpha^*);(p^*;\beta\alpha_1+(1-\beta)\alpha_2))}{s((p^*;\alpha^*);(p^*;\alpha_k))} \geq 1.$$

In the case of high aspiration level (proposition 3), the reasoning for type 1 is equivalent. For type 2, since all acts α and $\alpha' \in [0;1]$, fulfill

$$\lim_{t\to\infty} \frac{U_t^i(\alpha)}{U_t^i(\alpha')} = 1,$$

preference for diversification trivially obtains.■

Proof of corollary 6

The statement about type 1 obtains trivially from the proof of Corollary 5, as well as the statement that type 2 exhibit preferences for diversification as long as

$$\bar{u}^2 < \mu(1 \mid p^*(1)).$$

For the case $\bar{u}^2 > \mu(1 \mid p^*(1))$, observe that

$$\lim_{t\to\infty} \frac{U_t^2(1)}{U_t^2(0)} = 1,$$

whereas

$$\lim_{t\to\infty} \frac{U_t^2(\alpha)}{U_t^2(1)} < 1$$

for all $\alpha \in (0;1)$. Note that each such α can be written as

$$\alpha \cdot 1 + (1-\alpha) \cdot 0,$$

which completes the proof demonstrating that type 2 has no preference for diversification in this case.■

References

Allen, F., Gale, D. (1994). ”Limited Market Participation and Volatility of Asset Prices”, American Economic Review 84: 933-955. Aragones, E. (1997). ”Negativity Effect and the Emergence of Ideologies”, Journal of Theoretical Politics 9: 189-210.

Arrow, K. J. (1970). Essays in the Theory of Risk-Bearing, North-Holland Publishing Company, Amsterdam - London.

Arthur, W. B. (1995). ”Complexity in Economic and Financial Markets”, Complexity 1: 20-25.

Attanasio, O. P., Banks, J., Tanner, S. (2002). ”Assets Holding and Consumption Volatility”, Journal of Political Economy 110: 771-792.

Barber and Odean (2000), ”Trading is Hazardous to Your Wealth: The Common Stock Investment Performance of Individual Investors”, Journal of Finance 55: 773-806.

Barber, B. M., Odean, T. (2001 a). ”Boys will Be Boys: Gender, Overconfidence and Common Stock Investment”, Quarterly Journal of Economics 116: 261-292.

Barber, B. M., Odean, T. (2001 b). ”Online Investors: Do the Slow Die First?”, University of California, Davis.

Barberis, N., Shleifer, A., Vishny, R. (1998). ”A Model of Investor Sentiment”, Journal of Financial Economics 49: 307-343.

Blonski, M. (1999). ”Social Learning with Case-Based Decision”, Journal of Economic Behavior and Organization 38: 59-77.

Börgers, T., Sarin, R. (2000). ”Naive Reinforcement Learning with Endogenous Aspirations”, International Economic Review 41: 921-950.

Bossert, W., Pattanaik, P. K., Xu, Y. (2000). ”Choice Under Complete Uncertainty: Axiomatic Characterization of Some Decision Rules”, Economic Theory 16: 295-312.

Buschena, D. E., Zilberman, D. (1995). ”Performance of the Similarity Hypothesis Relative to Existing Models of Risky Choice”, Journal of Risk and Uncertainty 11: 233-263.

Buschena, D. E., Zilberman, D. (1999). ”Testing the Effects of Similarity on Risky Choice: Implications for Violations of Expected Utility”, Theory and Decision 46 (3): 253-280.

Camerer, C. (1989). ”Bubbles and Fads in Asset Prices”, Journal of Economic Surveys 3: 3-41.

Cao, H. H., Wang, T., Zhang, H. (2005). ”Model Uncertainty, Limited Market Participation and Asset Prices”, Review of Financial Studies 18: 1219-1250.

Cochrane, J. H. (2002). ”Stocks as Money: Convenience Yield and the Tech-Stock Bubble”, NBER Working Paper No. w8987.

Cutler, D. M., Poterba, J. M., Summers, L. H. (1990). ”Speculative Dynamics and the Role of

Feedback Traders”, American Economic Review 80: 63-68.

Daniel, K. D., Hirshleifer, D., Subrahmanyam, A. (2001). ”Mispricing, Covariance and Risk and the Cross Section of Security Returns”, Journal of Finance 56: 921-965.

De Long, J. B., Shleifer, A., Summers, L., Waldman, R. (1990 a). ”Noise Trader Risk in Financial Markets”, Journal of Finance 45: 375-395.

De Long, J. B., Shleifer, A., Summers, L., Waldmann, R. J. (1990 b). ”Positive Feedback Investment Strategies and Destabilizing Rational Speculation”, Journal of Finance 45: 375-395.

Gayer, G. (2003). ”Perception of Probabilities in Situations of Risk A Case Based Approach”, Tel Aviv University, mimeo.

Gervais, S., Odean, T. (2001). ”Learning to be Overconfident”, Review of Financial Studies 14: 1-27.

Gilboa, I., Pazgal, A. (2001). ”Cumulative Discrete Choice”, Marketing Letters12: 118-130.

Gilboa, I., Schmeidler, D. (1995). ”Case-Based Decision Theory”, Quarterly Journal of Economics 110: 605-639.

Gilboa, I., Schmeidler, D. (1997 a). ”Cumulative Utility Consumer Theory”, International Economic Review 38: 737-761.

Gilboa, I., Schmeidler, D. (1997 b). ”Act Similarity in Case-Based Decision Theory”, Economic Theory 9: 47-61.

Gilboa, I., Schmeidler, D. (2001 a). ”Reaction to Price Changes and Aspiration Level Adjustments”, Review of Economic Design 6: 215-223.

Guerdjikova, A. (2001 b). ”The Case-Based Decision Theory — An Application for the Financial Markets”, University of Heidelberg, mimeo.

Guerdjikova, A. (2003). ”On the Definition and Existence of an Equilibrium in a Financial Market with Case-Based Decision Makers”, University of Heidelberg, mimeo.

Guerdjikova, A. (2004 a). ”Evolution of Wealth and Asset Prices in an Economy with Case-Based Decision Makers”, Discussion Paper 04-49, SFB 504, University of Mannheim.

Guerdjikova (2004 b). ”Case-Based Decision Theory and Financial Markets”, Inaugural Dissertation, Faculty of Social Sciences and Economics, University of Heidelberg.

Guerdjikova (2005). ”A Note on Case-Based Optimization with a Non-Degenerate Similarity Function”, Cornell University, mimeo.

Hommes, C., Sonnemans, J., Tuinstra, J., van de Helden, H. (2002). ”Expectations and Bubbles in Experimental Asset Pricing Model”, Working Paper, University of Amsterdam.

Jahnke, H., Chwolka, A., Simons, D. (2001). ”Coordinating Demand and Capacity by Adaptive Decision Making. An Application of the Case-Based Decision Theory”, Discussion Paper 471, University of Bielefeld.

Keynes, J. M. (1936). The General Theory of Implyment, Interest and Money, Macmillan, London.

Kindleberger, C. (1978). Manias, Panics and Crashes, Basic Books, New York.

Krause, A. (2003). ”Herding without Following the Herd: The Dynamics of Case-based Decisions with Local Interactions”, mimeo, University of Bath.

Kruskal, J. B. (1964) (a). ”Multidimensional scaling by optimizing goodness of fit to a nonmetric hypothesis”, Psychometrika 29: 1-27.

Kruskal, J. B. (1964) (b). ”Nonmetric multidimensional scaling: k numerical method”, Psychometrika 29: 115-129.

Mankiw, N. G., Zeldes, S. (1991). ”The Consumption of Stockholders and Nonstockholders”, Journal of Financial Economics 29: 97-112.

Matsui, A. (2001). ”Expected Utility and Case-Based Reasoning”, Mathematical Social Sciences 39: 1-12.

Lamont, O., Thaler, R. H. (2002). ”Can the Market Add and Subtract? Mispricing in Tech Stock Carve-outs”, manuscript, University of Chicago.

Meyn, S. P., Tweedie, R. L. (1996). Markov Chains and Stochastic Stability, Springer Verlag, Berlin, Heidelberg.

Nehring, K., Puppe, C. (1999). ”A Theory of Diversity”, SFB Discussion Paper A-605, University of Bonn.

Odean, T. (1999). ”Do Investors Trade Too Much?”, American Economic Review 89: 1279-1298.

Oliven, K., Rietz, T. A. (1995). ”Suckers Are Born But Markets Are Made: Individual Rationality, Arbitrage and Market Efficiency on an Electronic Futures Market”, Working Paper, University of Iowa.

Pazgal, A. (1997). ”Satisficing Leads to Cooperation in Mutual Interests Games”, Journal of Game Theory 26: 439-453.

Rietz, T. A. (1998). ”Enforcing Arbitrage Restrictions in Experimental Markets”, Working Paper, University of Iowa.

Rosenthal, L., Young, C. (1990). ”The Seemingly Anomalous Price Behavior of Royal Dutch Shell and Unilever nv/plc”, Journal of Financial Economics 26: 123-141.

Shepard, R. N. (1987). ”Towards a Universal Law of Generalization for Psychological Science”, Science 237: 1317-1323.

Shepard, R. N. (1962) (a). ”The analysis of proximities: Multidimensional scaling with an unknowndistance function I”, Psychometrika 27: 125-140.

Shepard, R. N. (1962) (b). ”The analysis of proximities: Multidimensional scaling with an unknowndistance function II”, Psychometrika 27: 219-246.

Shiller, R. (1981). ”Do Stock Prices Move Too Much to Be Justified by Subsequent Changes in Dividends?”, American Economic Review 71: 421-436.

Shiller, R. J. (1990). ”Market Volatility and Investor Behavior”, American Economic Review 80: 58-62.

Shleifer, A. (2000). Inefficient Markets. An Introduction to Behavioral Finance, Oxford University Press, Oxford, New York.

Shleifer, A., Vishny, R. (1997). ”The Limits of Arbitrage”, Journal of Finance 52: 35-55.

Sunder, S. (1995). ”Experimental Asset Markets: A Survey” in: The Handbook of Experimental Economics, Kagel, J., Roth, A. (eds.), Princeton University Press, Princeton.

Vissing-Jorgssen, A. (2002). ”Limited Asset Market Participation and the Elasticity of Intertemporal Substitution”, Journal of Political Economy 110: 825-853.

Vissing-Jorgenssen, A., Attanasion, O.P. (2003). ”Stock-Market Participation, Intertemporal Substitution and Risk-Aversion”, AEA Papers and Proceedings 93: 383-391.

Williamson, S. (1994). ”Liquidity and Market Participation”, Journal of Economic Dynamics and Control 18: 629-670.

Zizzo, D. J. (2002). ”Preliminary Experimental Results on the Similarity Function in 2×2 and 3×3 Games”, University of Oxford, mimeo.